Weierstrass points on a tropical curve

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$$W(D_N) = \{x \in X : \exists H \subset \mathbb{P}^r \text{ s.t. } m_x(H \cap X) \geq r+1\}$$
$$= \left\{ x \in X : \begin{array}{c}\text{“higher-than-expected” tangency with} \\ \text{some hyperplane } H \text{ at } x\end{array} \right\}$$

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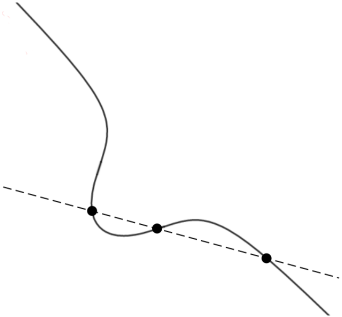

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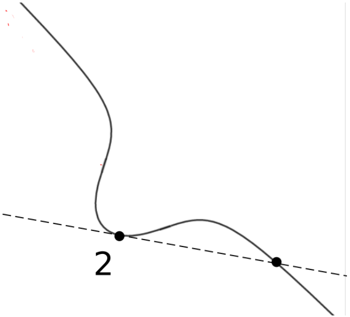

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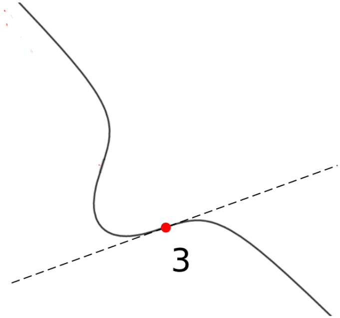

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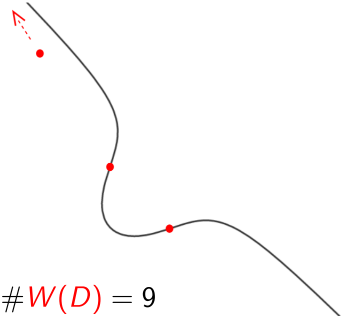

$N = 3$, $\quad \#W(D) = 9$

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N-torsion points on an elliptic curve	$\leftrightarrow$	Weierstrass points of D_N on a higher-genus curve

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Numerical "evidence": as N grows,

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Problem

How are Weierstrass points distributed on an algebraic curve?

Weierstrass points: genus 1, complex case

Problem

How are Weierstrass points distributed on genus 1 curve $X/\mathbb{C}$?

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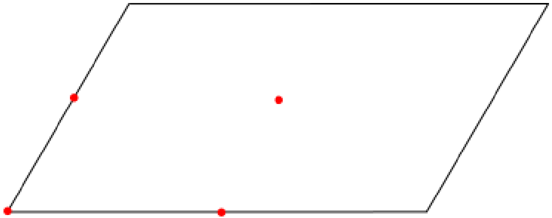

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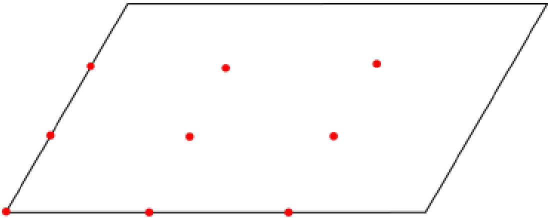

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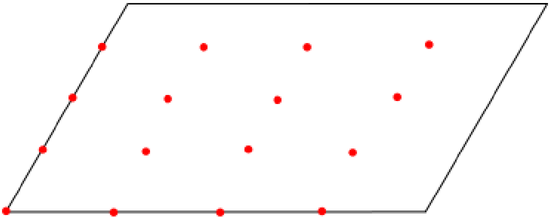

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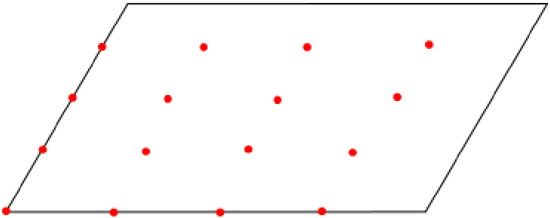

$\rightsquigarrow$ Weierstrass points distribute **uniformly**, w.r.t. $\mathbb{C} \to \mathbb{C}/\Lambda$

Weierstrass points: genus ≥ 2, complex case

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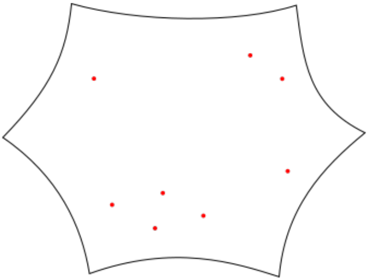

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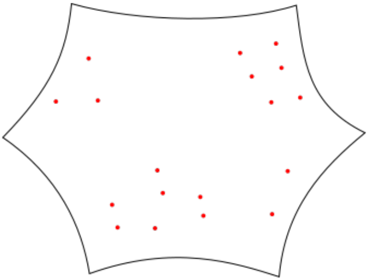

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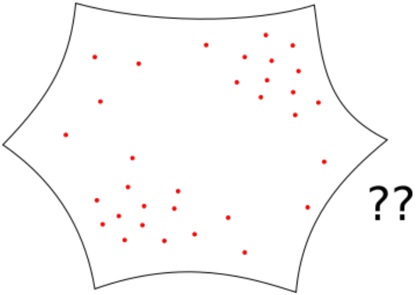

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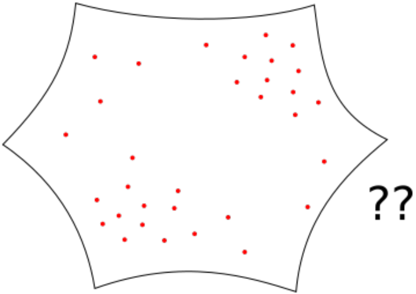


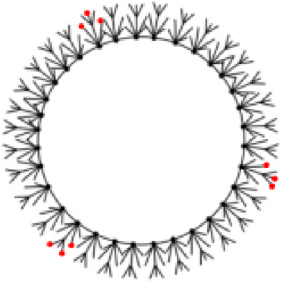
Theorem (Neeman, 1984)

Suppose X is a complex algebraic curve of genus $g \geq 2$. Then $W(D_N)$ distributes according to the Bergman measure as $N \to \infty$.

Weierstrass points: non-Archimedean case

Problem

How are Weierstrass points distributed on $X/\mathbb{K}$, *val* : $\mathbb{K}^\times \to \mathbb{R}$?



Source: Matt Baker's math blog

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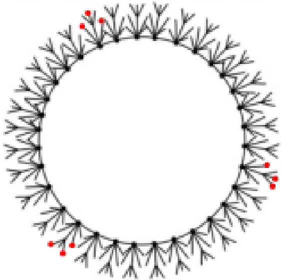
How are Weierstrass points distributed on $\cancel{X/\mathbb{K}}$ X^{an}?



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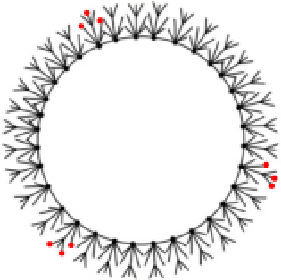

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Theorem (Amini, 2014)

Suppose X^{an} is a Berkovich curve of genus $g \geq 2$. Then $W(D_N)$ distributes according to the Zhang measure as $N \to \infty$.

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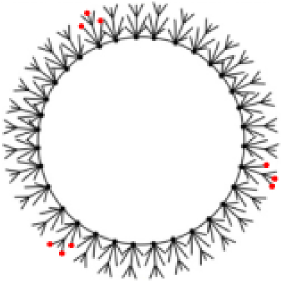

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Does the distribution follow from considering <u>only</u> the skeleton $\Gamma \subset X^{an}$?

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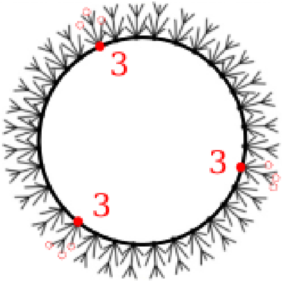

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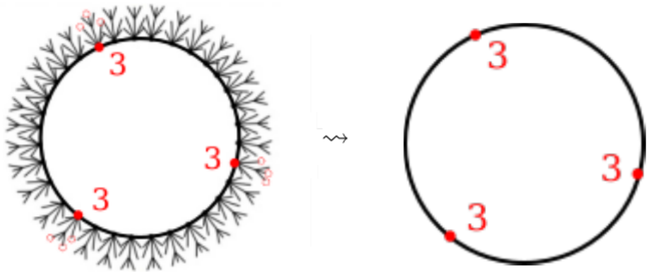
What is a tropical curve?

Tropical curve $\;(= \text{a skeleton of } X^{\mathrm{an}})$



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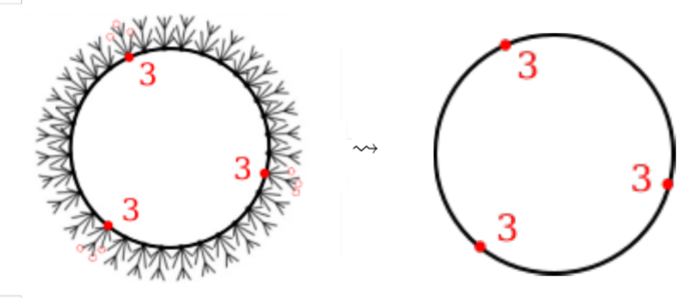
Tropical curve ($=$ a skeleton of X^{an})

(combinatorics) $=$ finite graph with edge lengths



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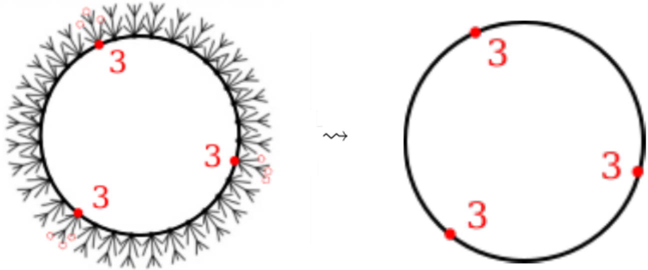

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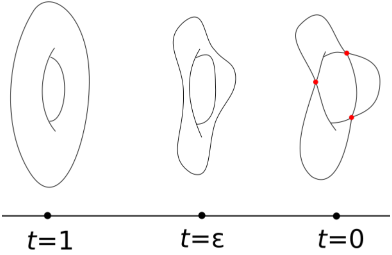

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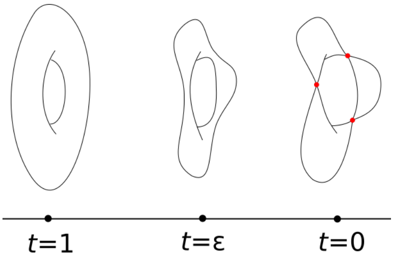

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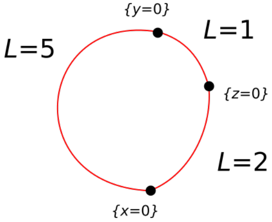

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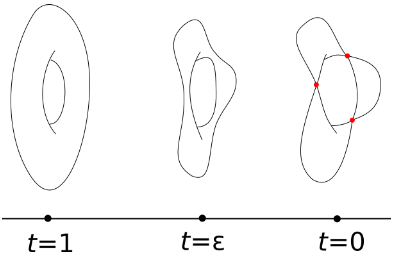

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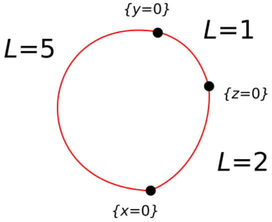

Tropical curves: divisor theory

Tropical curve = metric graph

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divisors $\mathrm{Div}(X)$	$\rightsquigarrow$	divisors $\mathrm{Div}(\Gamma)$
meromorphic functions	$\rightsquigarrow$	piecewise $\mathbb{Z}$-linear functions
linear system $\lvert D \rvert$	$\rightsquigarrow$	linear system $\lvert D \rvert$
$= \mathbb{P}^r$		$=$ polyhedral complex of dim $\geq r$
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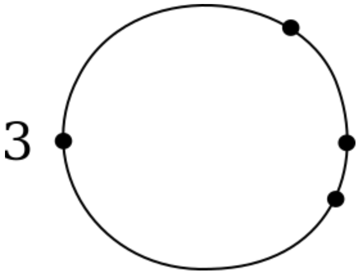

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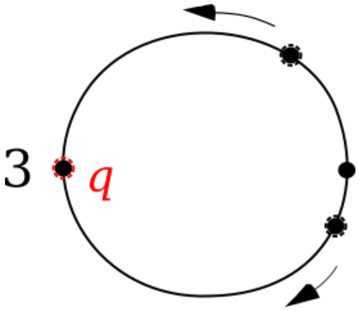

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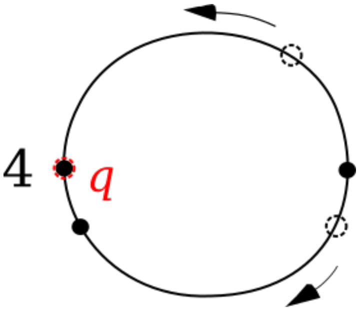

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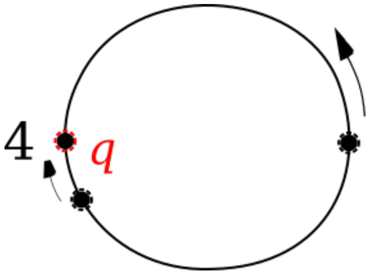

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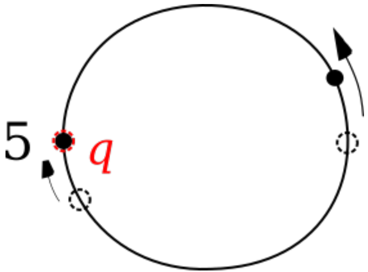

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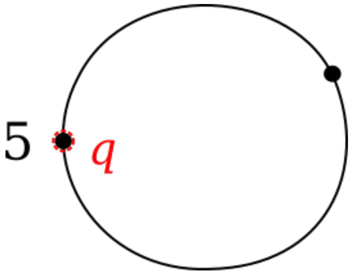

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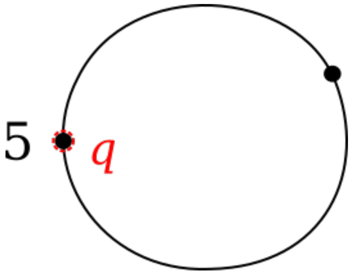

What happens as q varies?

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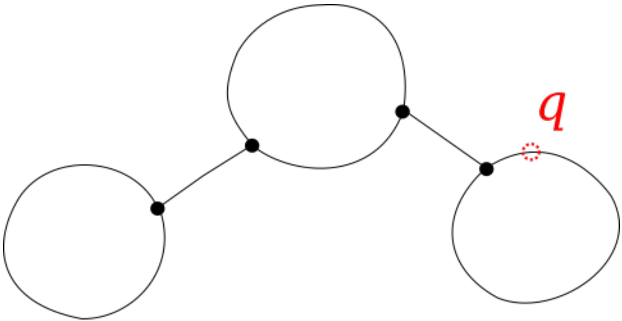

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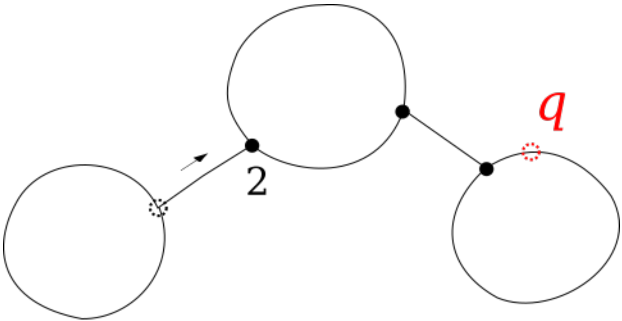

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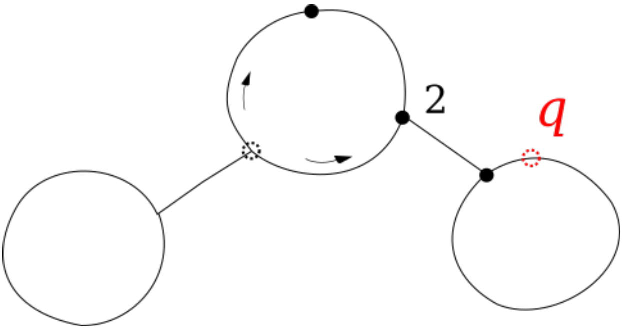

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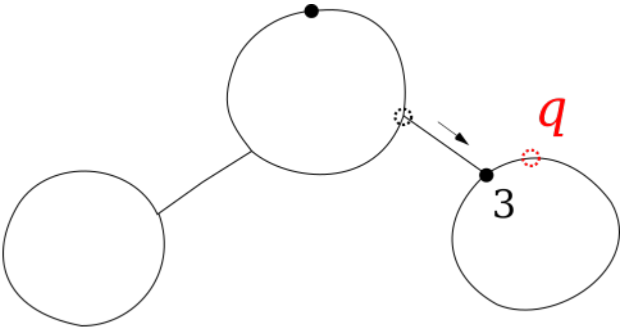

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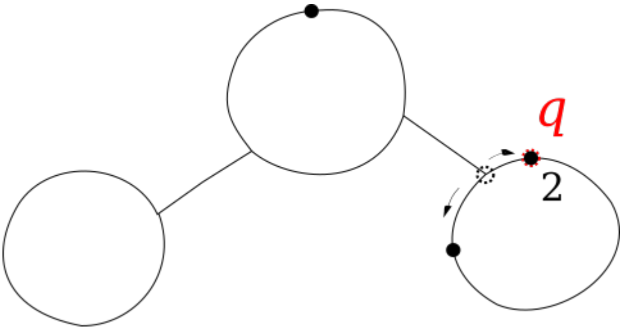

Tropical curves: Weierstrass points

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Definition: $\Gamma =$ metric graph, D_N divisor of degree N
$\rightsquigarrow$ Baker–Norine rank $r = r(D_N)$

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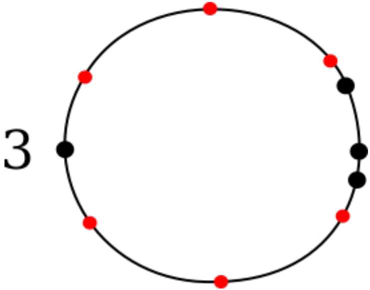
EXCEPT sometimes $\#(\text{Weierstrass points}) = \infty$

Weierstrass points: tropical case

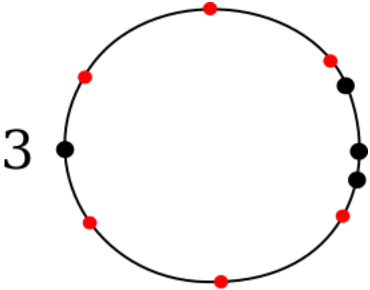
Example: Genus $g(\Gamma) = 1$:



degree $D = 6$,

Weierstrass points: tropical case

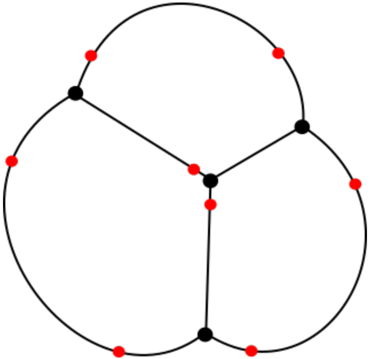
Example: Genus $g(\Gamma) = 1$:



degree $D = 6$, $\rightsquigarrow$ $\#(W(D)) = 6$

Weierstrass points: tropical case

Example: Genus $g(\Gamma) = 3$:



degree $D = 4$,

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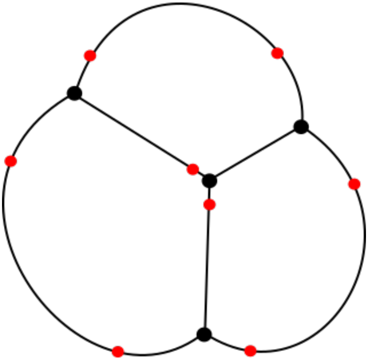

degree $D = 4$, $\rightsquigarrow$ $\#(W(D)) = 8$

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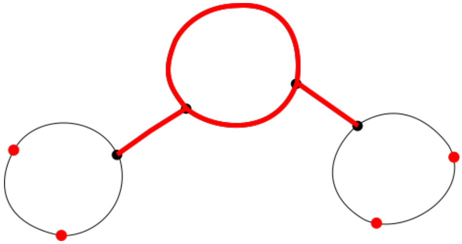

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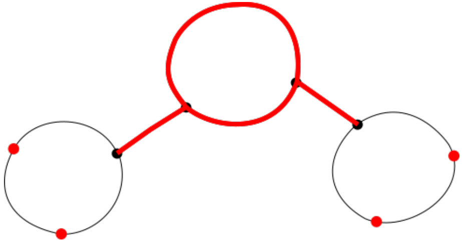

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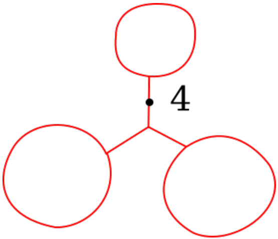

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In general, this problem doesn't happen!

Theorem (R)

For a generic divisor class $[D]$, the Weierstrass locus $W(D)$ is finite.

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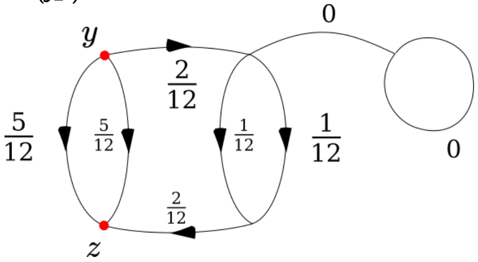

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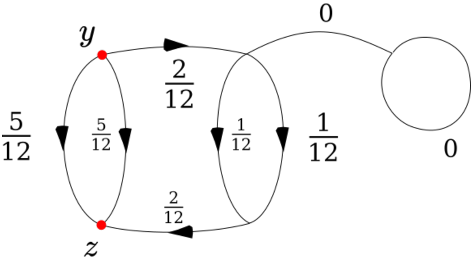
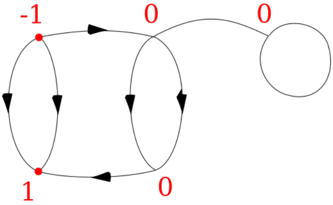


satisfies Laplacian $\Delta(j_z^y) = z - y$



Electrical networks: Canonical measure

$\Gamma =$ metric graph

Definition ("electrical" version, Chinburg–Rumely–Baker–Faber)

Zhang's **canonical measure** μ on an edge is the "current defect"

$$\begin{aligned}\mu(e) &= \text{current bypassing } e \text{ when 1 unit sent from } e^- \text{ to } e^+ \\ &= 1 - (\text{current through } e \text{ when } \ldots\ )\end{aligned}$$

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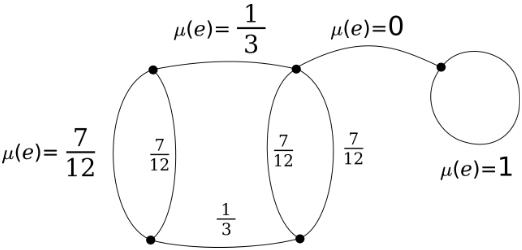

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Generally:

- $0 \leq \mu(e) \leq 1$
- $\mu(e) = 0 \Leftrightarrow e$ a bridge
- $\mu(e) = 1 \Leftrightarrow e$ a loop

Foster's Theorem: $\mu(\Gamma) = \sum_{e \in E} \mu(e) = g$

Tropical Weierstrass distribution: proof idea

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Namely, for any edge e

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$$\begin{array}{ccc} \text{(discrete current flow)} & \xrightarrow{N \to \infty} & \text{(continuous current flow)} \\ \updownarrow & & \updownarrow \\ \#(\text{Weierstrass points on } e) & & \text{canonical measure } \mu(e) \end{array}$$

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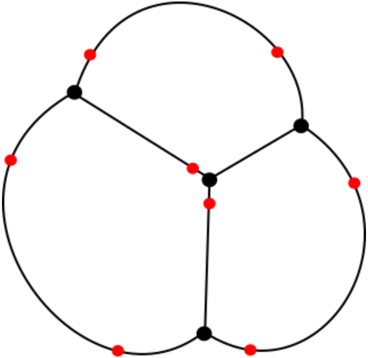
Weierstrass points on a tropical curve



Thank you!