Uniform bounds on tropical torsion points

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Harry Richman

TGIF Seminar

21 January 2022

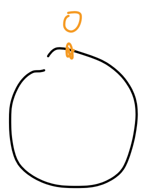
What are torsion points?

A = abelian group, n-torsion subgroup $A[n] = \{a \in A : n \cdot a = 0\}$

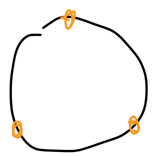
torsion subgroup $A_{tors} = \bigcup_{n \geq 1} A[n]$

<u>Ex</u>. $A = \mathbb{R}/\mathbb{Z}$

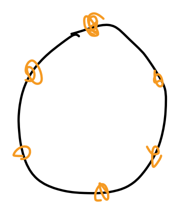
$A[1]$



$A[3]$



$A[6]$



<u>Ex</u>. $A = \mathbb{R}^2/\mathbb{Z}^2$

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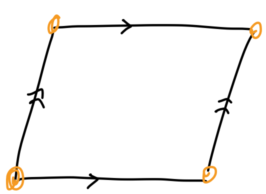
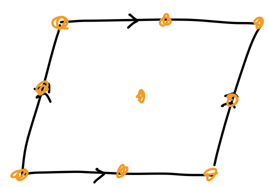


$A[2]$



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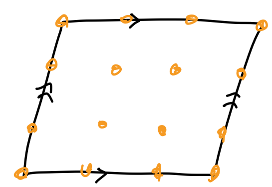

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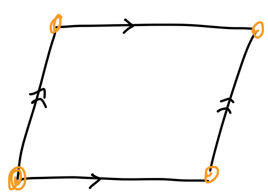

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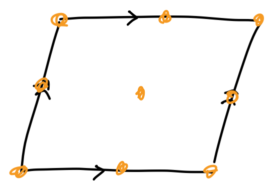
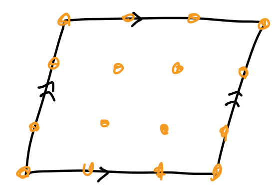


$A[3]$



For this talk: always have

$A = \mathrm{Jac}(X)$ or $A = \mathrm{Jac}(\Gamma)$ $\Rightarrow$ $\boxed{A_{tors} \cong \mathbb{Q}^n / \mathbb{Z}^n}$

$\downarrow$ algebraic curve $\qquad$ $\downarrow$ tropical curve

$\cong \mathbb{R}^{2g} / \mathbb{Z}^{2g}$ $\qquad$ $\cong \mathbb{R}^g / \mathbb{Z}^g$

Why care about torsion points?

- Start with rational points on varieties, $X(\mathbb{Q}) = X \cap \mathbb{Q}^n$ (in affine space)

<u>Fermat Conjecture</u> (Wiles et. al) If $n \geq 3$,

$$\#\{\text{solutions to } x^n + y^n = z^n \text{ in } \mathbb{Q}^3\}/\text{scaling} < \infty$$

≤ 4 ?

<u>Mordell Conjecture</u> (Faltings, 1983)

X = alg. curve of genus ≥ 2, $\#X(\mathbb{Q}) < \infty$

<u>Uniform Mordell Conjecture</u> (Open)

X = alg. curve of genus $g \geq 2$ $\#X(\mathbb{Q}) \leq N(g)$

Why care about torsion points?

- Apply analogy:

rational points on X $\rightsquigarrow$ torsion points in Jacobian

$\mathbb{Q}^n \cap X$ $\qquad$ $\underbrace{\mathrm{Jac}(X)_{\mathrm{tors}}}_{= \mathbb{Q}^n/\mathbb{Z}^n} \cap X$ $\rightarrow$ using embedding $\iota_q : X \hookrightarrow \mathrm{Jac}(X)$

Mordell Conjecture (Faltings, 1983)

X = alg. curve of genus ≥ 2,

$$\# X(\mathbb{Q}) < \infty$$

Manin - Mumford Conjecture (Raynaud, 1983)

X = alg. curve of genus ≥ 2,

$$\# \left(\iota_q(X) \cap \mathrm{Jac}(X)_{\mathrm{tors}} \right) < \infty$$

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Uniform Manin - Mumford Conj. (Kühne, Looper - Silverman - Wilmes)

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Why care about torsion points?

Apply analogy:

rational points on X $\rightsquigarrow$ torsion points in Jacobian

$\mathbb{Q}^n \cap X$ $\qquad$ $\mathrm{Jac}(X)_{\mathrm{tors}} \cap X$

Apply another analogy:

algebraic curve $\rightsquigarrow$ tropical curve

X $\qquad$ Γ

$\mathrm{Jac}(X)$ $\qquad$ $\mathrm{Jac}(\Gamma)$

Trop. <u>Manin - Mumford</u> <u>Conjecture</u> (Raynaud, 1983)

Γ = ~~alg.~~ trop. curve of genus ≥ 2,

$$\#\left(\iota_q(\Gamma) \cap \mathrm{Jac}(\Gamma)_{\mathrm{tors}}\right) < \infty$$

Trop. <u>Uniform Manin - Mumford</u> <u>Conj.</u> (Kühne, Looper - Silverman - Wilmes)

Γ = ~~alg.~~ trop. curve of genus $g \geq 2$,

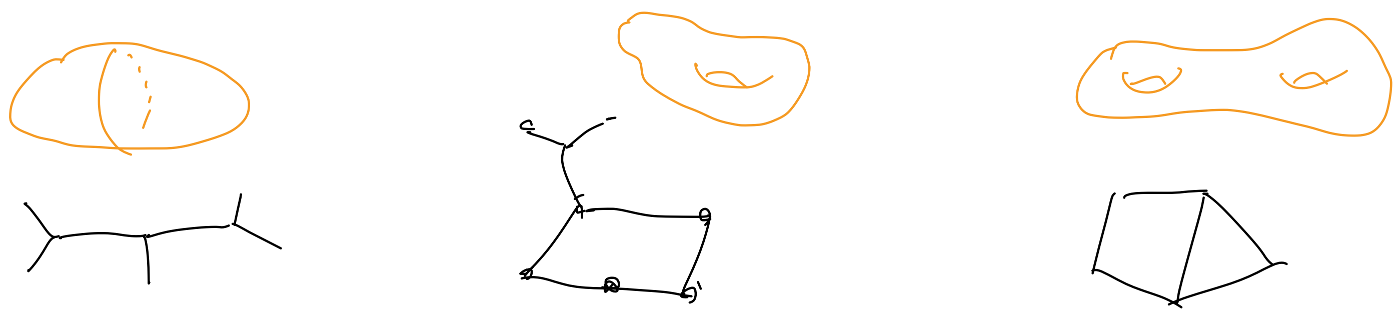
$$\#\left(\iota_q(\Gamma) \cap \mathrm{Jac}(\Gamma)_{\mathrm{tors}}\right) \leq N(g)$$

Tropical curves = metric graph

$\Gamma = (G, \ell)$ where $G = (V, E)$ finite, connected graph

$\ell : E \to \mathbb{R}_{>0}$ length function on edges

Ex.



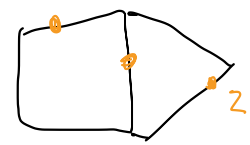
$g = 0$ $g = 1$ $g = 2$

The genus of Γ is $g = \dim H_1(\Gamma, \mathbb{R})$

Tropical curves : Divisors & Jacobian

A divisor on Γ is a formal $\mathbb{Z}$-sum of points in Γ

Ex. $D = x + y + 2z$



A divisor is effective if all coeffs. are ≥ 0.

The degree of a divisor is sum of coeffs.

$$\deg\Big(\sum_{x \in \Gamma} a_x \cdot x \Big) = \sum_{x \in \Gamma} a_x$$

a_x in $\mathbb{Z}$

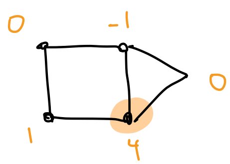
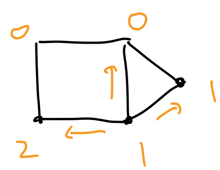
The Jacobian of Γ

$$\operatorname{Jac}(\Gamma) \overset{\text{def}}{=} \operatorname{Pic}^0(\Gamma) = \begin{pmatrix} \text{degree } 0 \text{ divisors} \\ \text{on } \Gamma \end{pmatrix} \Big/ \begin{pmatrix} \text{tropical} \\ \text{linear equivalence} \end{pmatrix}$$

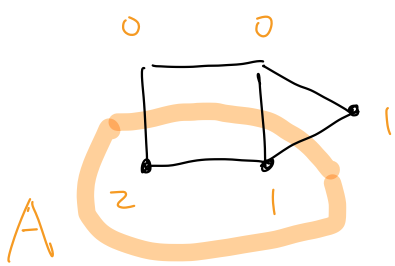
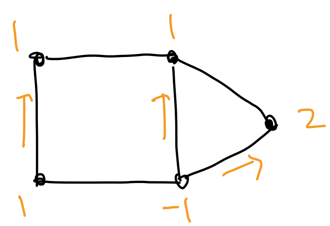
Linear equivalence : Discrete case

$G = (V, E)$ graph   i.e. unit edge lengths

Equivalence relation generated by "firing" moves

Ex.  $\sim$ 

data = choose induced subgraph $A \subset G$

Ex.  $\sim$ 

Linear equivalence : Continuous case

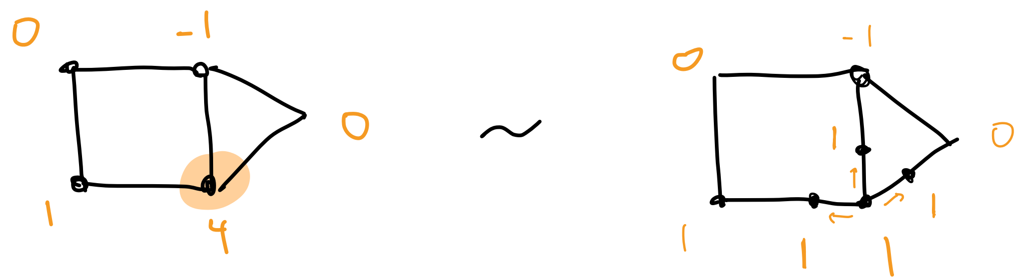
$\Gamma = (G, \ell)$ arbitrary edge lengths $\mathbb{R}_{>0}$

Equivalence relation generated by "continuous-firing" moves, data: (A, ε)

closed subset of Γ

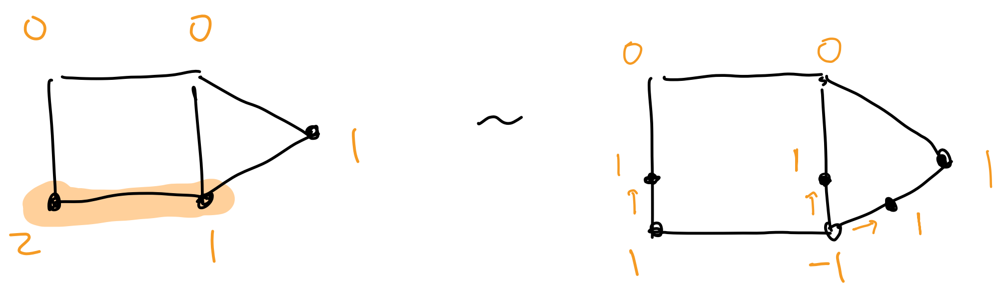
$\mathbb{R}_{>0}$

Ex.



moving chips move same distance on all edges

Ex.



OR

Tropical curves and Jacobians

$\Gamma = (G, \ell)$ tropical curve, $\quad Jac(\Gamma) = Div^0(\Gamma) / (\text{linear equivalence})$

Abel - Jacobi embedding: choose $q \in \Gamma$

$$\begin{aligned} \iota_q : \Gamma &\longrightarrow Jac(\Gamma) \\ x &\longmapsto [x - q] \end{aligned}$$

Theorem (Mikhalkin - Zarkhov) If Γ has genus g,

$$Jac(\Gamma) \cong \mathbb{R}^g / \mathbb{Z}^g \quad \Rightarrow \quad Jac(\Gamma)_{tors} \cong \mathbb{Q}^g / \mathbb{Z}^g$$

$\downarrow$

$H^1(\Gamma, \mathbb{R})^\vee / H_1(\Gamma, \mathbb{Z})$

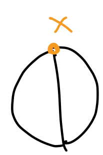
Tropical curves and Jacobians

$\Gamma = (G, \ell)$ tropical curve, $\quad \mathrm{Jac}(\Gamma) = \mathrm{Div}^0(\Gamma) \,/\, (\text{linear equivalence})$

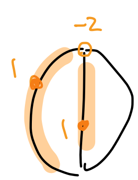
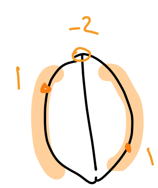
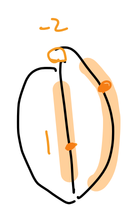
Theorem (An - Baker - Kuperberg - Shokrieh)

Up to linear equivalence, a divisor class $[D]$ of deg. 0 has a unique* representative whose positive support lies in an edge set of G whose complement is a spanning tree.

* up to choosing bscpt.

Ex. $\Gamma =$  , three types* of divisor classes in $\mathrm{Jac}(\Gamma)$:

× = basepoint

$\{$  ,  ,  $\}$

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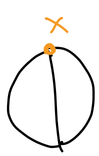
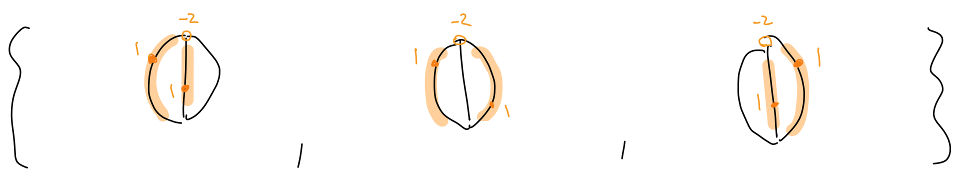
Ex. $\Gamma =$ 

$\times$ = basepoint

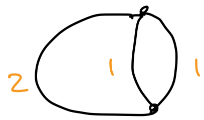
, three types* of divisor classes in $Jac(\Gamma)$:



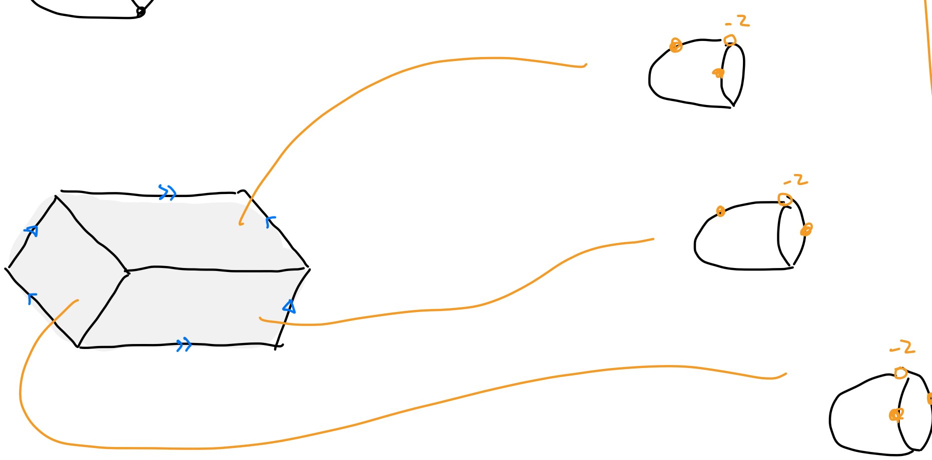
Theorem $\Rightarrow$ $Jac(\Gamma)$ decomposes as union of cells indexed by spanning trees of $\Gamma = (G, \ell)$

Tropical curves and Jacobians

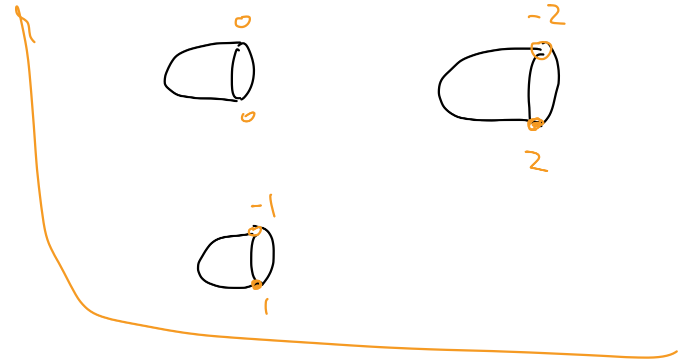
Ex. Γ =



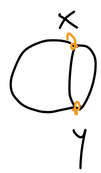
$\text{Jac}(\Gamma)$ =

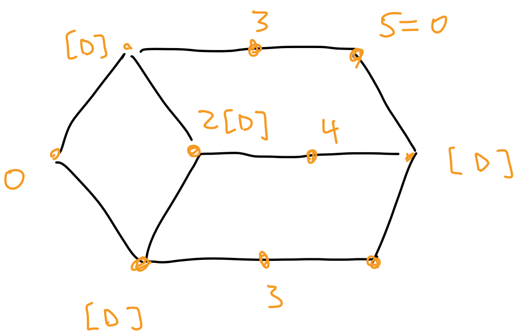


On Boundaries:



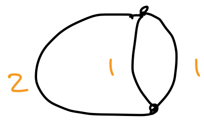
Multiples of $[x-y]$ when



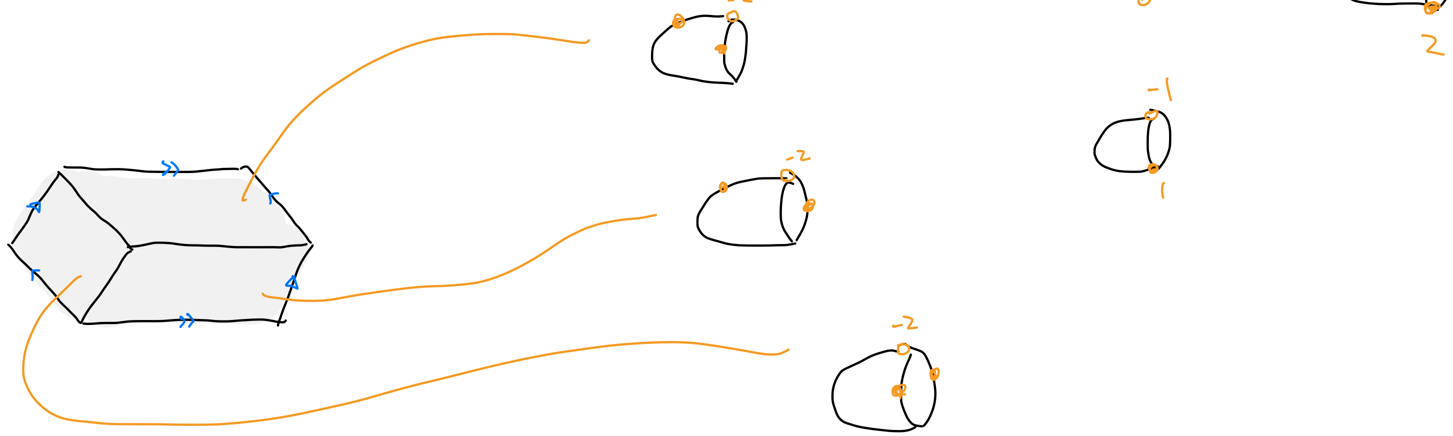


Tropical curves and Jacobians

Ex. Γ =



$\mathrm{Jac}(\Gamma)$ =



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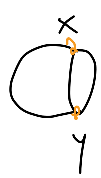
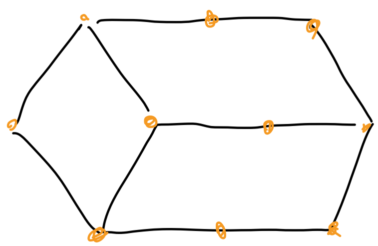
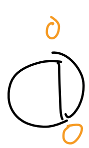




$\Rightarrow$ $[x-y]$ is 5-torsion

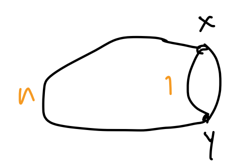
i.e.

$\sim$



Exercise On tropical curve

Γ =

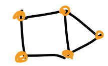


,

divisor class $[x-y]$ is $(2n+1)$-torsion

Tropical torsion points: Failure of finite bounds

Fact: In a graph Γ w/ unit edge lengths, all vertices are torsion points

Ex. 

↪ Vertex-supported divisors form "critical group" of G.

↪ #(critical gp) = #(spanning trees of G)

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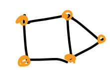
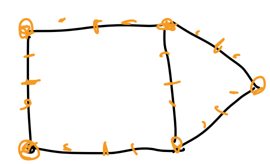
Ex. 

↪ Vertex-supported divisors form "critical group" of G.

rational, by rescaling & subdividing

Fact: In a graph Γ w/ ~~unit~~ edge lengths, there are ∞-many torsion points, i.e. $\#\left(c_q(\Gamma) \cap \mathrm{Jac}(\Gamma)_{\mathrm{tors}}\right) = \infty$

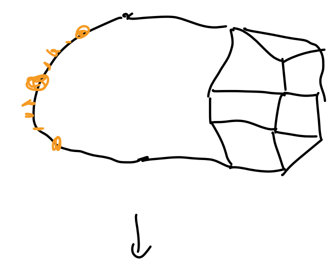
Ex. 

$\Rightarrow$ Tropical "Manin-Mumford Conjecture" fails

Tropical torsion points: Failure of finite bounds

Fact: In a graph Γ w/ arbitrary edge lengths, if a single edge contains $\mathbb{Z}$ torsion points, then it contains ∞-many

Ex. $\Gamma =$



$\mathbb{R}^g/\mathbb{Z}^g \supset \mathbb{Q}^g/\mathbb{Z}^g$

Justification: Abel-Jacobi embedding $\iota_q : \Gamma \to \mathrm{Jac}(\Gamma)$ is affine on each edge of Γ

$\Rightarrow$ Tropical "Manin-Mumford Conjecture" really fails, i.e. locally

Tropical torsion points : Results

Theorem (R.) [Conditional uniform tropical Manin-Mumford]

For a metric graph Γ of genus g, if the numbers of torsion points is finite then

$$\#\left(\iota_q(\Gamma) \cap \mathrm{Jac}(\Gamma)_{\mathrm{tors}} \right) \leq \boxed{3g-3}$$

bound

← # edges in G

Theorem (R.) [General tropical Manin-Mumford]

If G is a biconnected graph of genus $g \geq 2$, then $\Gamma := (G, \ell)$ has finitely many torsion points for very general edge lengths $\ell : E(G) \to \mathbb{R}_{>0}$

Tropical torsion points : Results

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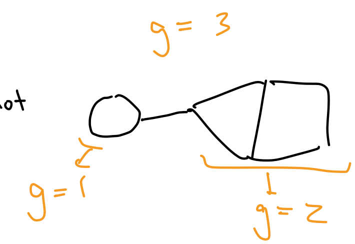
If G is a biconnected graph of genus $g \geq 2$, then $\Gamma = (G, \ell)$ has finitely many torsion points for very general edge lengths $\{\ell : E(G) \to \mathbb{R}_{>0}\} \cong \mathbb{R}^{\#E}_{>0}$

- "biconnected" ensures that Γ "behaves like" genus ≥ 2 Ex. not



- "very general" in $\mathbb{R}^n$ means away from countable collection of positive-codim. algebraic subsets,

Ex. $U_1 = \mathbb{R}^n \setminus \{ (x_1, \dots, x_n)$ where some $x_i \in \mathbb{Q} \}$

Ex. $U_2 = \mathbb{R}^n \setminus \{ (x_1, \dots, x_n)$ where $f(x_1, \dots, x_n) = 0$ for some polynomial w/ $\mathbb{Q}$ coeffs. $\}$

Tropical torsion points : Results

Theorem (R.) [General tropical Manin - Mumford]

If G is a biconnected graph of genus $g \geq 2$, then $\Gamma = (G, \ell)$ has finitely many torsion points for very general edge lengths $\ell : E(G) \to \mathbb{R}_{>0}$

Proof Idea:

- Torsion condition on $[x-y]$ equivalent to rational slopes on "unit potential function" $j_y^x : \Gamma \to \mathbb{R}$

- Kirchhoff : Each slope of j_y^x is ratio of $\mathbb{Z}$-polynomial of edge lengths $\ell : E \to \mathbb{R}_{>0}$

- $\{ f(x_1, \ldots, x_n) \notin \mathbb{Q}$ for $\mathbb{Z}$-polynomials $f \}$ forms countable collection

Tropical torsion points : Higher degree

Higher-degree Abel-Jacobi embedding, choose $Q \in \mathrm{Div}^d(\Gamma)$

$$\iota^{(d)}_{[Q]} : \Gamma \times \cdots \times \Gamma \longrightarrow \mathrm{Jac}(\Gamma)$$

$$(x_1, \ldots, x_d) \longmapsto [x_1 + \cdots + x_d - Q]$$

Problem: When is

$$\#\left(\iota^{(d)}_{[Q]}(\Gamma^d) \cap \mathrm{Jac}(\Gamma)_{\mathrm{tors}} \right) < \infty \ ?$$

Tropical torsion points : Higher degree

Higher-degree Abel-Jacobi embedding, choose $Q \in \mathrm{Div}^d(\Gamma)$

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<u>Problem</u>: When is

$$\#\left(\iota_{[Q]}^{(d)}(\Gamma^d) \cap \mathrm{Jac}(\Gamma)_{\mathrm{tors}} \right) < \infty \; ?$$

<u>Theorem</u> (R.) [Conditional uniform higher-degree ..]

If it is finite, then

$$\#\left(\iota_{[Q]}^{(d)}(\Gamma^d) \cap \mathrm{Jac}(\Gamma)_{\mathrm{tors}} \right) \leq \binom{3g+3}{d}$$

Tropical torsion points: Higher degree

Higher-degree Abel-Jacobi embedding, choose $Q \in \mathrm{Div}^d(\Gamma)$

$$\iota^{(d)}_{[Q]} : \Gamma \times \cdots \times \Gamma \longrightarrow \mathrm{Jac}(\Gamma)$$

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Problem: When is

$$\#\left(\iota^{(d)}_{[Q]}(\Gamma^d) \cap \mathrm{Jac}(\Gamma)_{\mathrm{tors}} \right) < \infty \ ?$$

Theorem (R.)

If G has independent girth $\gamma^{\mathrm{ind}}(G) \leq d$, then it is not finite.

Otherwise, if $\gamma^{\mathrm{ind}}(G) > d$ then it is finite for $\Gamma = (G, \ell)$ for very general edge lengths $\ell : E(G) \to \mathbb{R}_{>0}$

[Note: $\gamma^{\mathrm{ind}}(G) \geq 2 \iff$ biconnected components have $g \geq 2$]

Independent girth

$G = (V, E)$

Recall girth is length of shortest cycle

$$\gamma(G) = \min_{C \in \mathcal{C}(G)} \{ \# E(C) \}, \qquad \mathcal{C}(G) = \{\text{all cycles of } G\}$$

Let $rk^{\perp} : E(G) \to \mathbb{Z}$ denote rank of cographic matroid, i.e.

$$rk^{\perp}(A) = \# A + \underbrace{1 - h_0(G \setminus A)}_{\leq 0} \quad \text{for } A \subset E$$

Def'n The independent girth of G is

$$\gamma^{ind}(G) = \min_{C \in \mathcal{C}(G)} \{ rk^{\perp}(E(C)) \} \leq \gamma(G)$$

→ Where has this been studied?

[Thanks!]

Uniform bounds on tropical torsion points

arXiv: 2112.00168

Harry Richman

TGiF Seminar

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