# Weierstrass points on tropical curves

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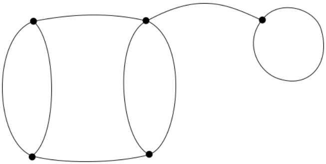
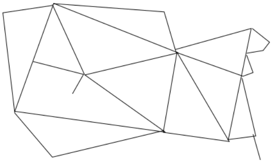
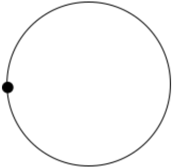
# Introduction

A **metric graph** is a network made of vertices and edges, where each edge has a fixed length.







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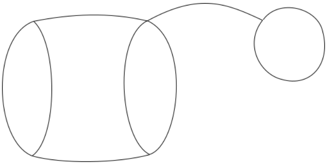
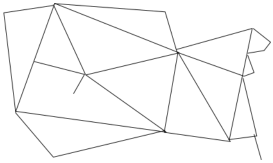
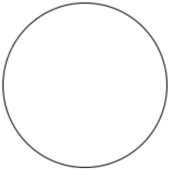




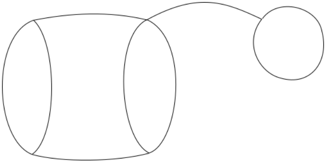


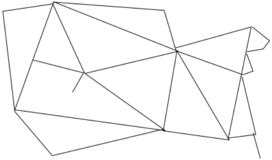
The **genus** of a metric graph is the number of “independent cycles”.

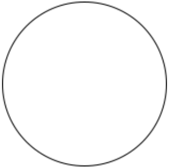
# Introduction

## Problem

How to place $N$ points on a metric graph in an "evenly spaced" way?







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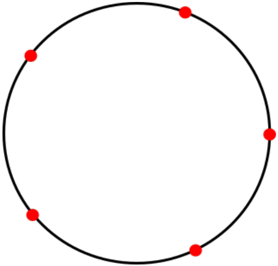

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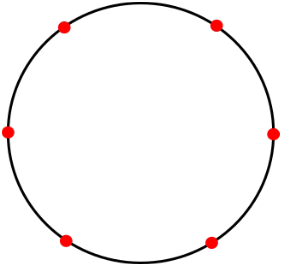

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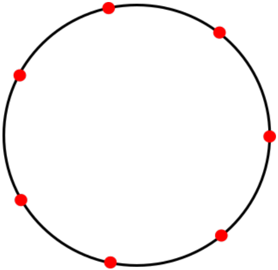

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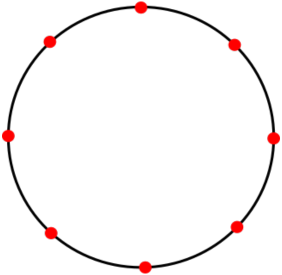

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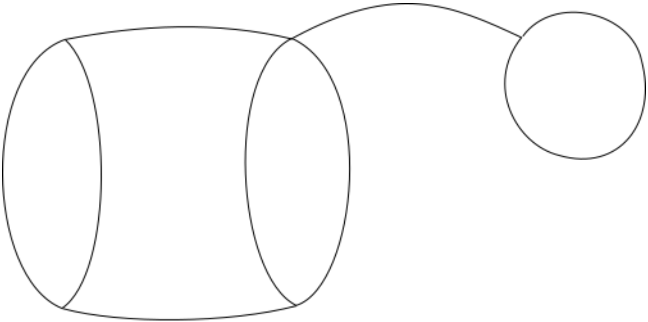

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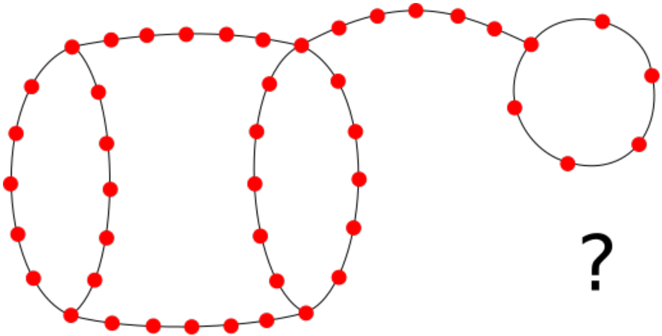

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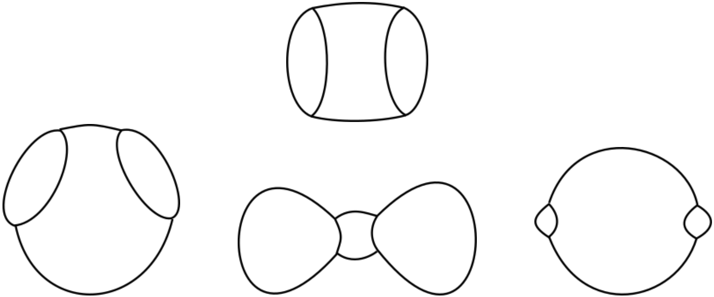
How to place $N$ points on a metric graph in an "evenly spaced" way?

Objectives:
- capture global topological structure
- ignore "dead ends"



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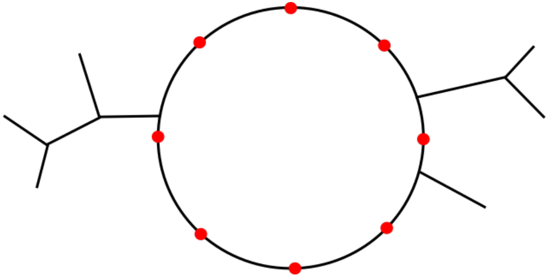
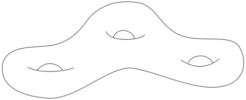
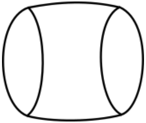


# Introduction: What is tropical geometry?

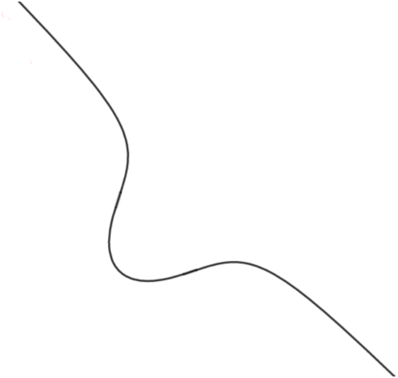
connection between algebraic geometry and combinatorics

| algebraic geometry | combinatorics |
|---|---|
| Riemann surface | metric graph |
|  |  |
| polynomials | piecewise-linear functions |

# What is algebraic geometry?

Study polynomial equations and their solution sets.

Example: $x^3 + y^3 + xy + 1 = 0, \qquad x, y \in \mathbb{R}$



Source: desmos.com

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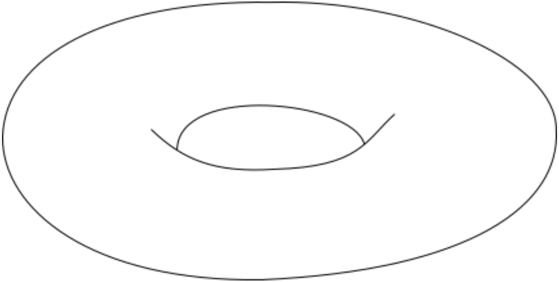

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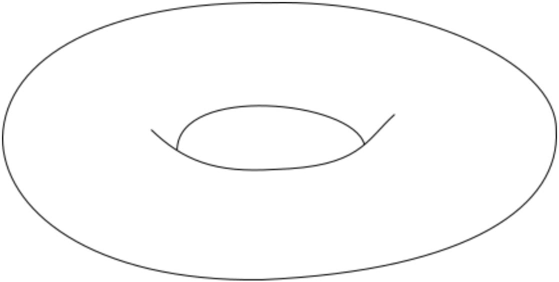

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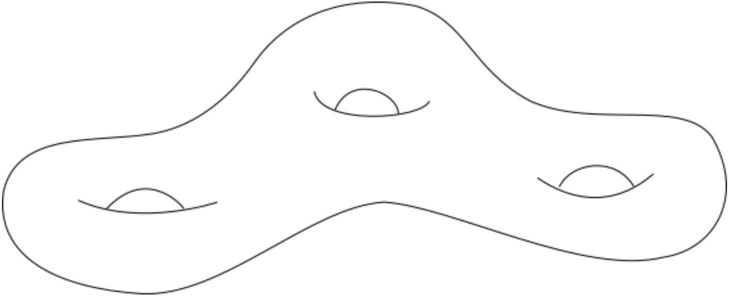
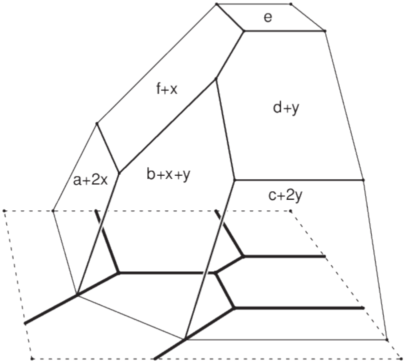


The solution set in $\mathbb{C}$ is a **Riemann surface**.
The **genus** is the number of "holes".

# What is tropical geometry?

Study piecewise-linear functions ("tropical polynomials")
and their break loci ("solution sets").

Example: $\min\{a + 2x, b + x + y, c + 2y, d + y, \ldots\}$




Source: Richter-Gebert, Sturmfels, Theobald

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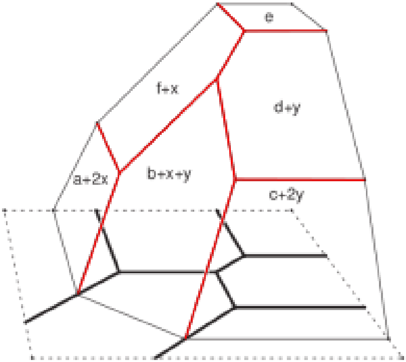


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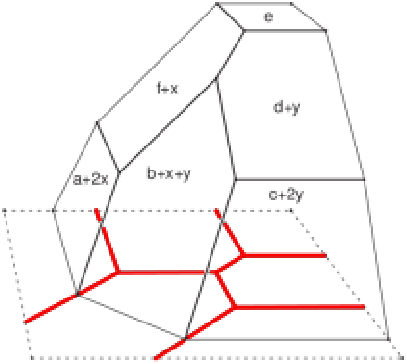


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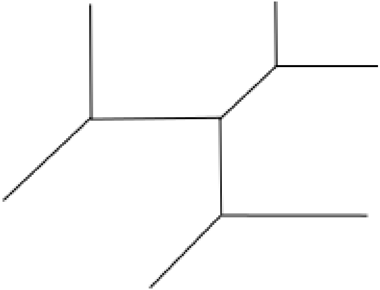

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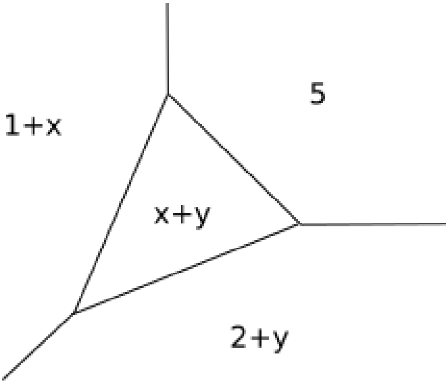
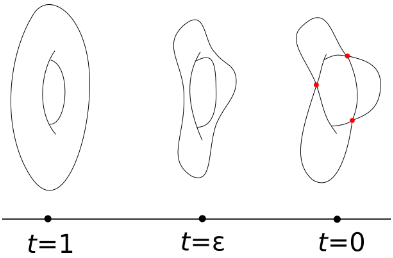
Example: $\min\{1 + 3x, 2 + 3y, x + y, 5\}$



# Tropicalizing algebraic curves

Turn a Riemann surface into a metric graph via a degenerating family.

Example: $X_t = \{tx^3 + t^2y^3 + t^5z^3 + xyz = 0\} \subset \mathbb{P}^2_{\mathbb{C}}$



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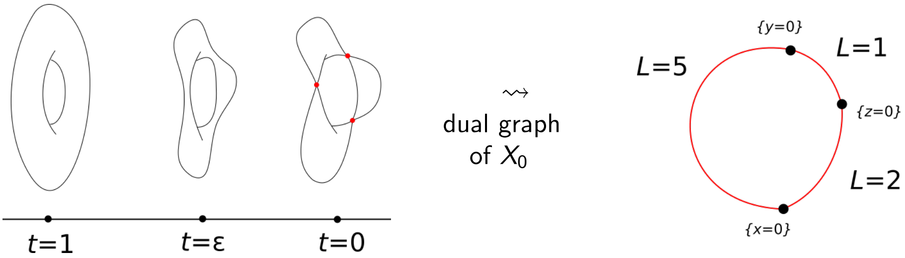

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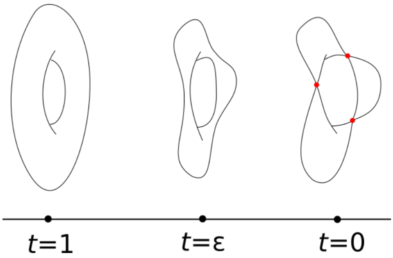
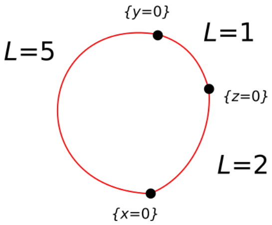



$\rightsquigarrow$
dual graph of $X_0$



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$$X_t = \{tx^3 + t^2y^3 + t^5z^3 + xyz = 0\}, \qquad t,\ t^2,\ t^5 \in \mathbb{C}[t]$$

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$$X_t = \{a(t)x^3 + b(t)y^3 + c(t)z^3 + xyz = 0\}, \qquad a(t) \in \mathbb{C}((t))$$

$\mathbb{C}((t))$ = field of Laurent series, e.g. $a(t) = t + 2t^2 + 6t^3 + \cdots$

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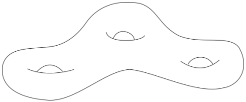
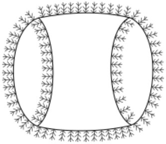
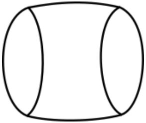
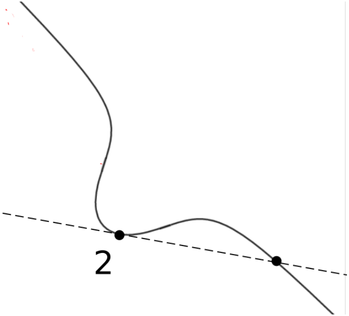
| algebraic geometry | non-Archimedean geometry | combinatorics |
| --- | --- | --- |
| Riemann surface | Berkovich curve | metric graph |
|  |  |  |
| polyn. over $\mathbb{C}$ | polyn. over $\mathbb{C}((t))$ | piecewise-linear func. |

# What is a Weierstrass point?

Idea: point whose tangent line has "higher-than-expected" tangency

Example: $X = \{x^3 + y^3 + xy + 1 = 0\}$



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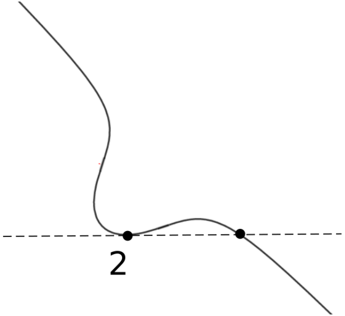

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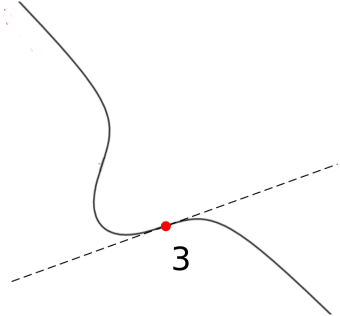

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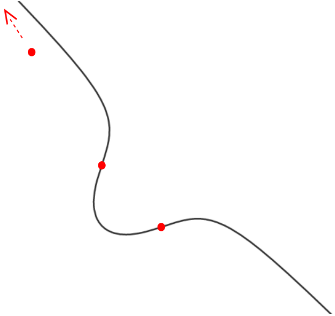

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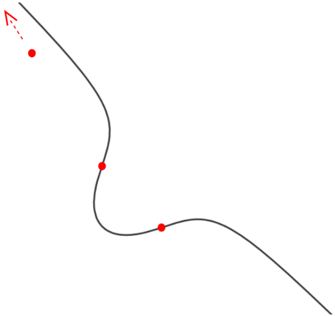
Example: $X = \{x^3 + y^3 + xy + 1 = 0\} \subset \mathbb{P}^2$



Generalize to high-dimensional embedding $X \to \mathbb{P}^r$

(projective embedding $\phi : X \to \mathbb{P}^r$) $\leftrightsquigarrow$ (linear equivalence class of $D$)

# Algebraic geometry: linear equivalence

A **divisor** is a finite collection of points, $D = p_1 + \cdots + p_N$.

The **degree** is the number of points, $N$.

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$$f(z) = c(z - z_i)^m + (\text{higher powers of } z - z_i).$$

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Divisors $D_1, D_2$ are **linearly equivalent** if there is a rational function $f(z)$ and constants $a_1, a_2$ such that

$$D_1 = \sum_{f(z_i)=a_1} ord_{z_i}(f(z) - a_1) \cdot z_i, \quad D_2 = \sum_{f(z_i)=a_2} ord_{z_i}(f(z) - a_2) \cdot z_i$$

Idea: $D_1 \sim D_2$ are different "level sets" of the same function

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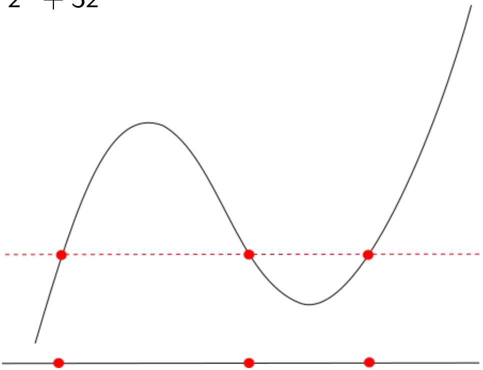
Example: $f(z) = z^3 + 3z^2$



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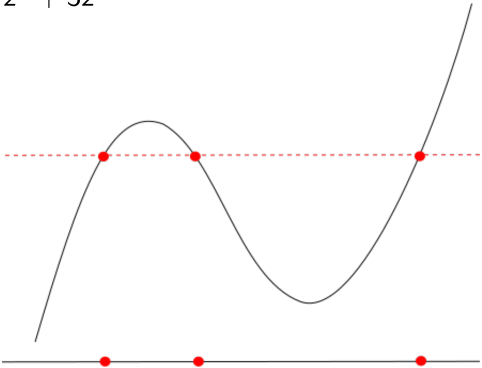

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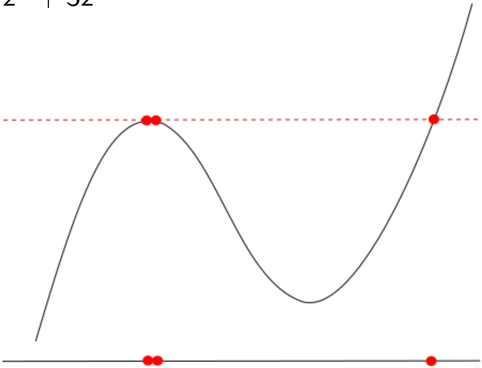

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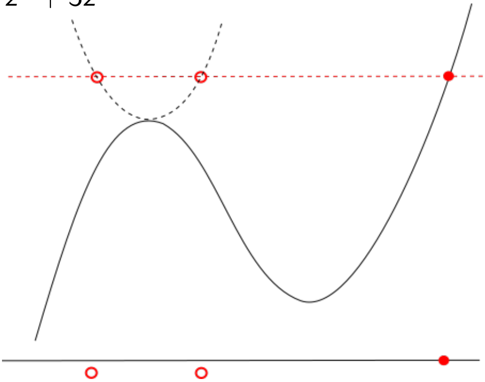

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$X$ a smooth algebraic curve, $D = p_1 + \cdots + p_N$ a divisor

$$\text{(linear equivalence class of } D) \leftrightsquigarrow \text{(projective embedding } \phi : X \to \mathbb{P}^r)$$

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Definition:

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"higher-than-expected" tangency with some hyperplane at $p$

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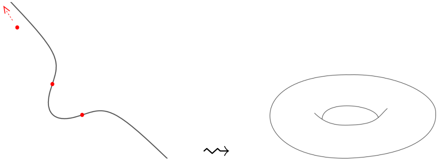
<u>Example</u>: $X$ = genus 1 curve over $\mathbb{C}$

$$\{x^3 + y^3 + xy + 1 = 0\}$$



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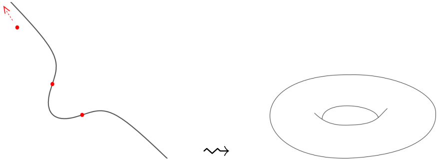

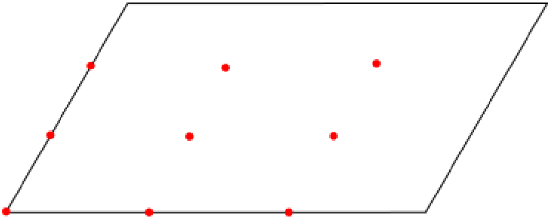

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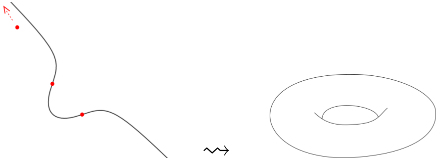

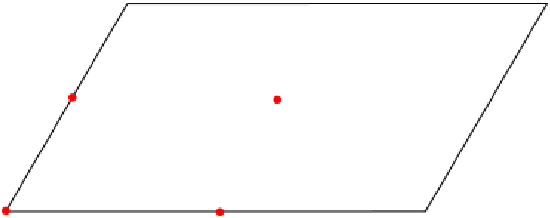

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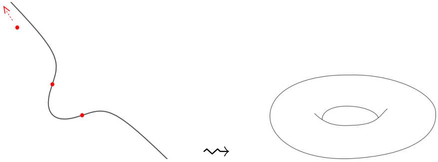

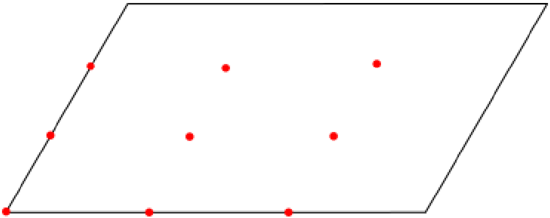

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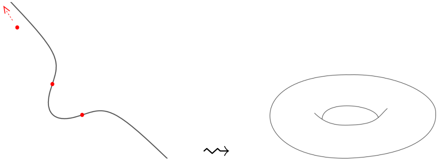

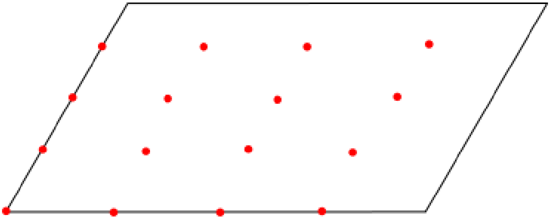

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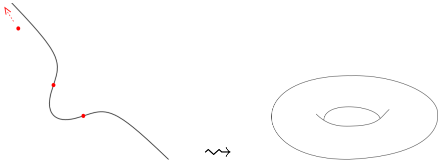

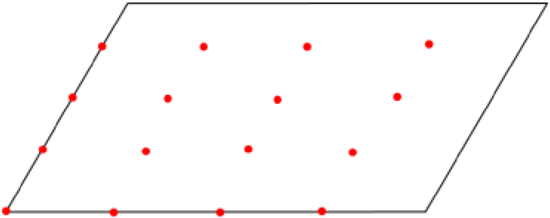

$\rightsquigarrow$ Weierstrass points are **evenly spaced** (w.r.t. addition law)

# What is a Weierstrass point?

Intuition (Mumford):

$$\begin{array}{c} N\text{-torsion points} \\ \text{on an elliptic curve} \end{array} \quad \leftrightarrow \quad \begin{array}{c} \text{Weierstrass points of } D_N \\ \text{on a higher-genus curve} \end{array}$$

Numerical "evidence": as $N$ grows,

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**Problem**

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# Weierstrass points: complex case

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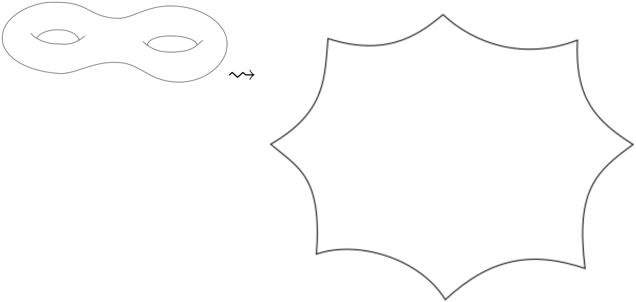
How are Weierstrass points distributed on higher genus curve $X/\mathbb{C}$?



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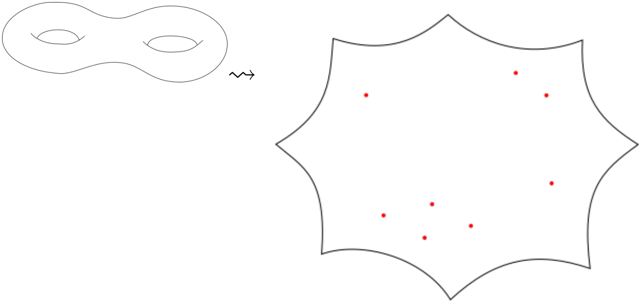

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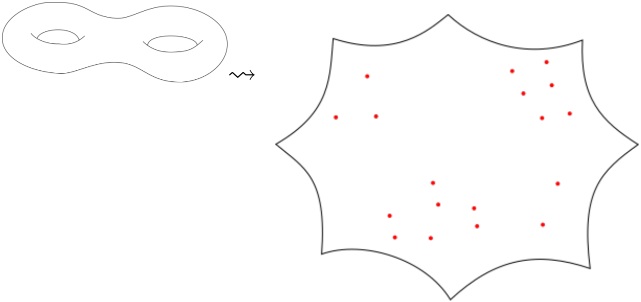

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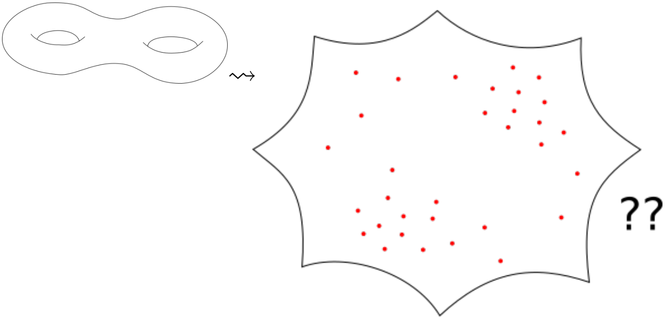

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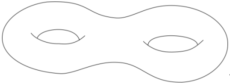
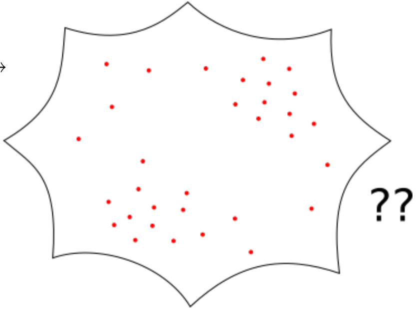


$\rightsquigarrow$



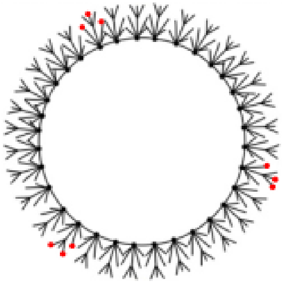

**Theorem (Neeman, 1984)**

*Suppose $X$ is a complex algebraic curve of genus $g \geq 2$. Then $W(D_N)$ distributes according to the Bergman measure as $N \to \infty$.*

# Weierstrass points: non-Archimedean case

## Problem

How are Weierstrass points distributed on $X^{an}/\mathbb{C}((t))$?



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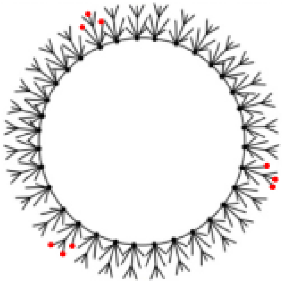

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## Theorem (Amini, 2014)

*Suppose $X^{an}$ is a Berkovich curve of genus $g \geq 2$. Then $W(D_N)$ distributes according to the Zhang measure as $N \to \infty$.*

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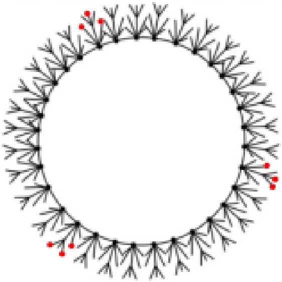

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Does the distribution follow from considering only the skeleton $\Gamma \subset X^{\mathrm{an}}$?

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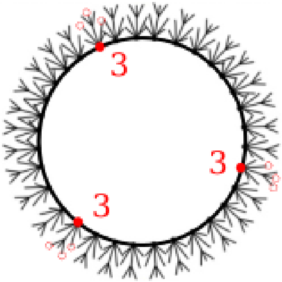


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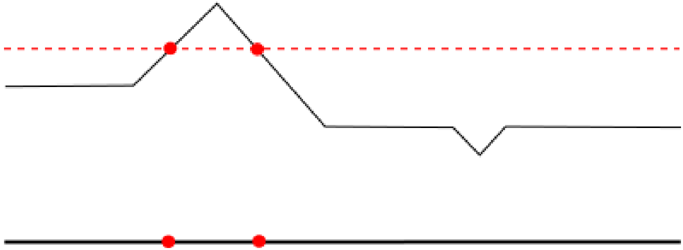
# Tropical linear equivalence

Study piecewise-$\mathbb{Z}$-linear function and its break locus ("solution set")

Idea: $D_1 \sim D_2$ for level sets of the same piecewise-linear function

Example:
$f(x) = \min\{0, x-4, 2x-7, 4x-9\} - \min\{0, 2x-7, 3x-9, 4x-9.5\}$



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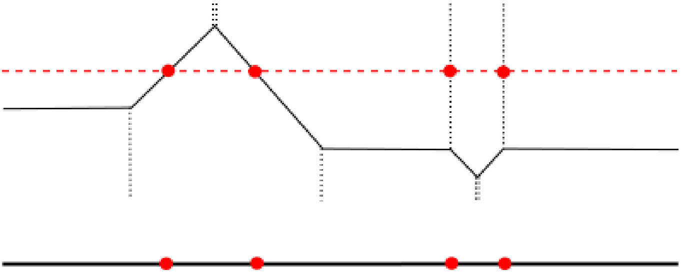

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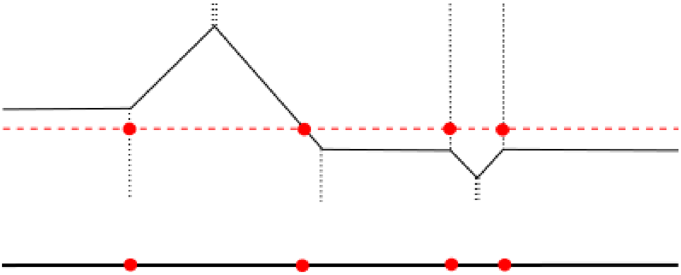

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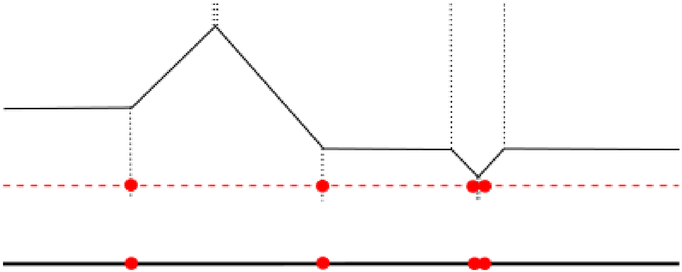

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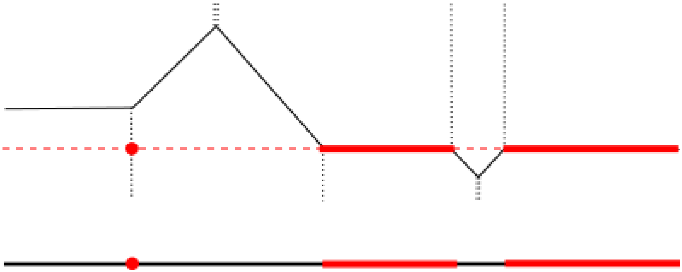

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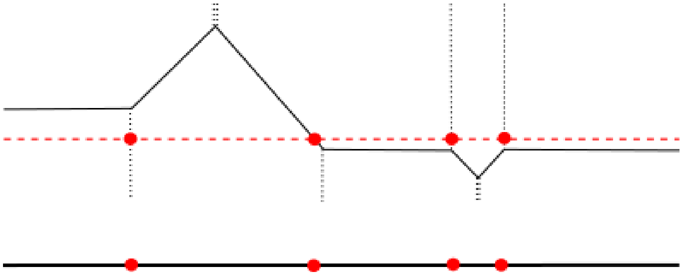

# Weierstrass points: tropical case

**Problem**

How are Weierstrass points distributed on a tropical curve?

$\Gamma$ a metric graph, $D = p_1 + \cdots + p_N$ a divisor

Definition:

$$W(D) = \{p \in \Gamma : D \sim (r+1)p + E \quad \text{for some } E\}$$

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$$\text{EXCEPT sometimes } \#(\text{Weierstrass points}) = \infty$$

# Tropical curves: reduced divisors

How to compute Weierstrass locus?

$$W(D) = \{q \in \Gamma : D \sim (r+1)q + E \quad \text{for some } E\}$$

# Tropical curves: reduced divisors

How to compute Weierstrass locus?

$$\begin{aligned} W(D) &= \{q \in \Gamma : D \sim (r+1)q + E \quad \text{for some } E\} \\ &= \{q \in \Gamma : \mathrm{red}_q[D] \geq (r+1)q\} \end{aligned}$$

Intuition: linear equivalence on $\Gamma$ = "discrete current flow"
$q$-**reduced** divisor $\mathrm{red}_q[D]$ = "energy-minimizing" divisor $\sim D$

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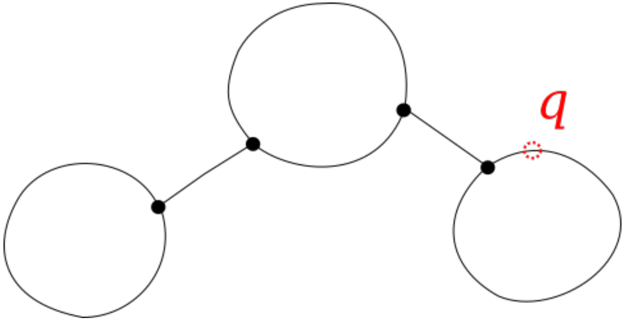

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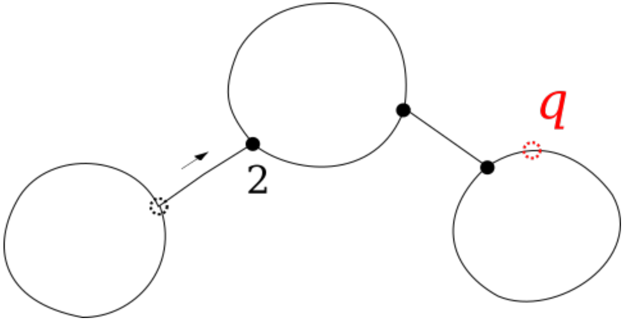

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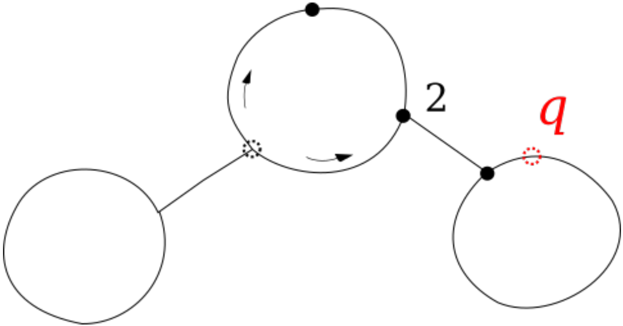

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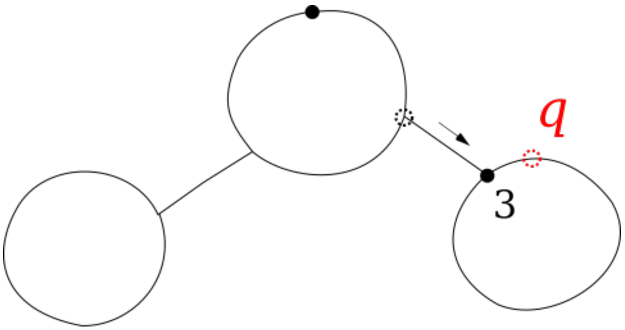

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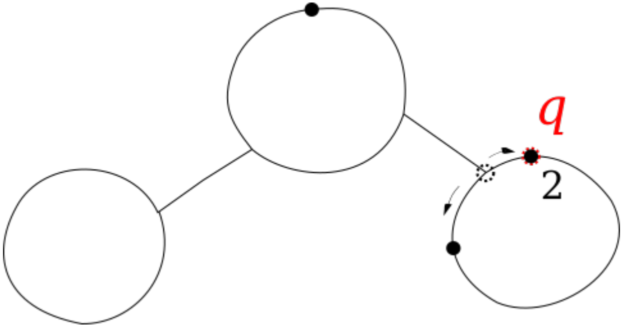

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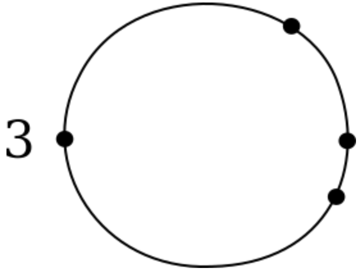

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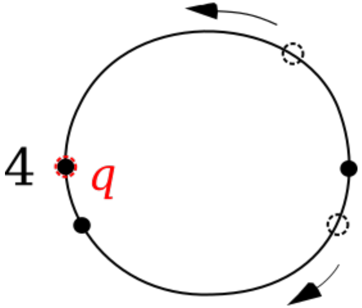

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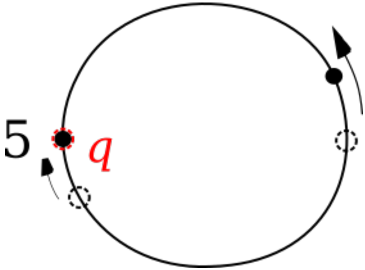

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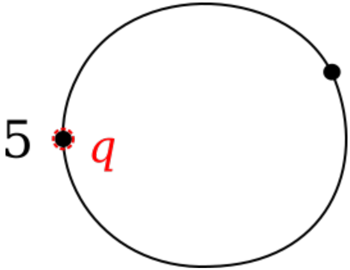
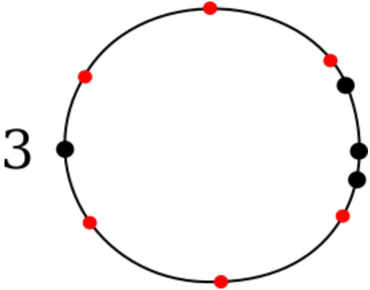



What happens as $q$ varies?

# Weierstrass points: tropical case

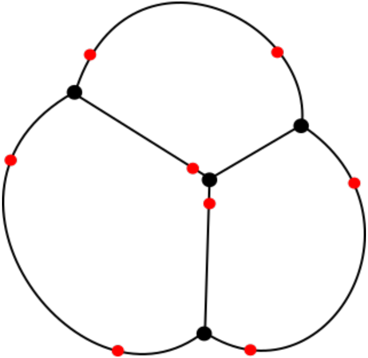
Example: Genus $g(\Gamma) = 1$, degree $D = 6$:




$$\#(W(D)) = 6$$

# Weierstrass points: tropical case

Example: Genus $g(\Gamma) = 3$, degree $D = 4$:



$\rightsquigarrow \#(W(D)) = 8$

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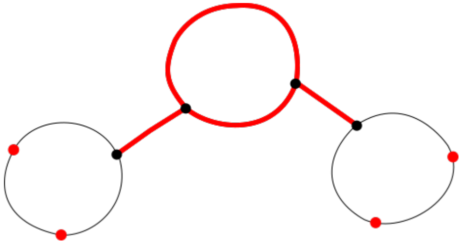

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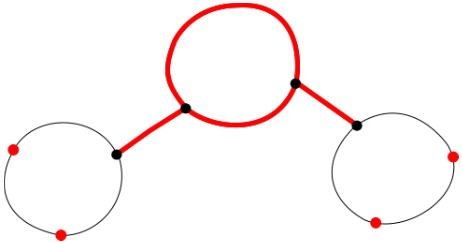

$$\#(W(D)) = \infty!$$

In general, this problem doesn't happen.

**Theorem (R)**

*For a generic divisor class $[D]$, the Weierstrass locus $W(D)$ is finite.*

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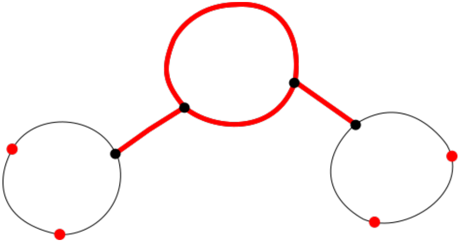

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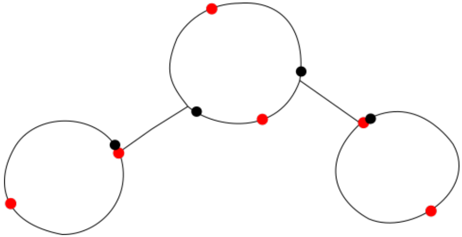

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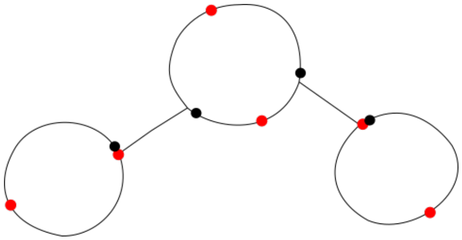

So, we can still ask about distribution of $W(D)$ "generically"

**Theorem (R)**

*For a sequence of generic divisor classes $[D_N]$ on $\Gamma$, the Weierstrass locus $W(D_N)$ distributes according to Zhang's canonical measure $\mu$.*

# Electrical networks

$\Gamma$ = electrical network by replacing each edge $\rightsquigarrow$ resistor

Given $y, z \in \Gamma$, let $$j_z^y = \left( \begin{array}{l} \text{voltage on } \Gamma \text{ when 1 unit of} \\ \text{current is sent from } y \text{ to } z \end{array} \right)$$

Observation: voltage function is piecewise-linear

By Ohm's law, current $= \frac{\text{voltage}}{\text{resistance}} =$ **slope** of $j_z^y$

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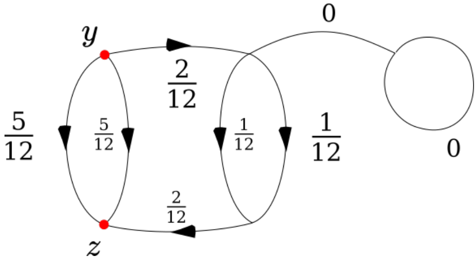
Example: current $= (j_z^y)'$



# Electrical networks: canonical measure

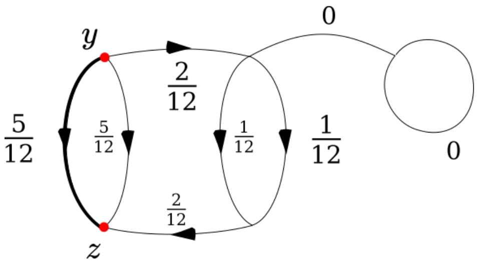
$\Gamma$ = metric graph

**Definition ("electrical" version, Chinburg–Rumely–Baker–Faber)**

Zhang's **canonical measure** $\mu$ on an edge is the "current defect"

$$\mu(e) = \text{current bypassing } e \text{ when 1 unit sent from } e^- \text{ to } e^+$$
$$= 1 - (\text{current through } e \text{ when } \dots)$$

Example:



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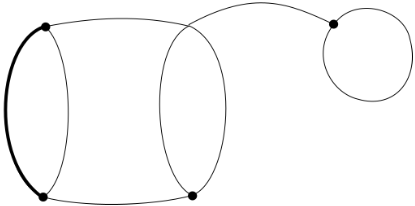
<u>Example</u>:

$\mu(e) = \frac{7}{12}$



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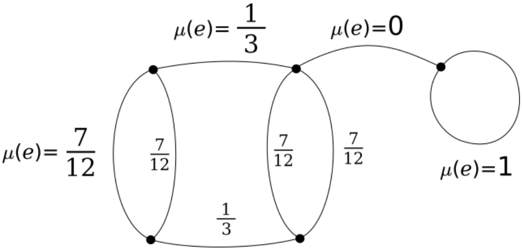

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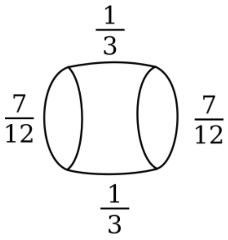

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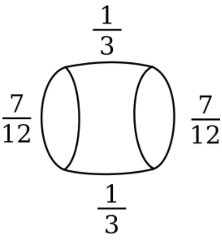


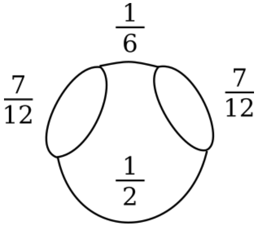


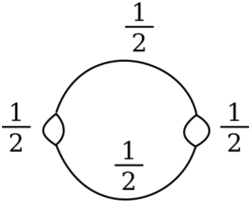


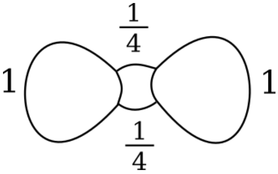

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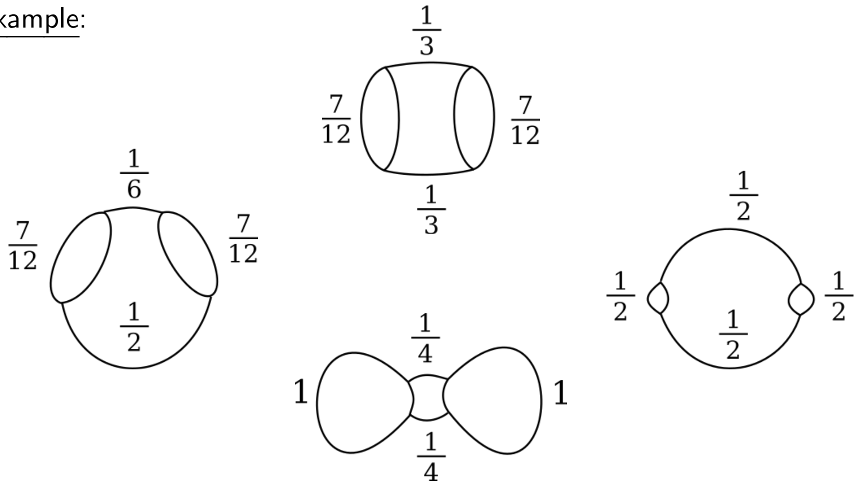


Foster's Theorem: $\mu(\Gamma) = \sum_{e \in E} \mu(e) = g$

# Tropical Weierstrass distribution: proof idea

**Theorem (R)**

*For a sequence of generic divisor classes* $[D_N]$ *on* $\Gamma$*, the Weierstrass locus* $W(D_N)$ *distributes according to Zhang's canonical measure* $\mu$*.*

*Namely, for any edge e*

$$\frac{\#(W(D_N) \cap e)}{N} \to \mu(e) \qquad \text{as} \qquad N \to \infty.$$

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Idea:

(continuous current flow)

$\updownarrow$

canonical measure $\mu(e)$

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$$\begin{array}{ccc} \text{(discrete current flow)} & \xrightarrow{N\to\infty} & \text{(continuous current flow)} \\ \updownarrow & & \updownarrow \\ \#(\text{Weierstrass points on } e) & & \text{canonical measure } \mu(e) \end{array}$$

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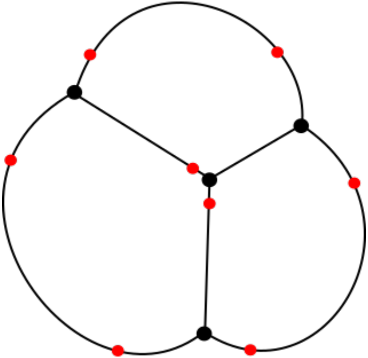

# Thank you!