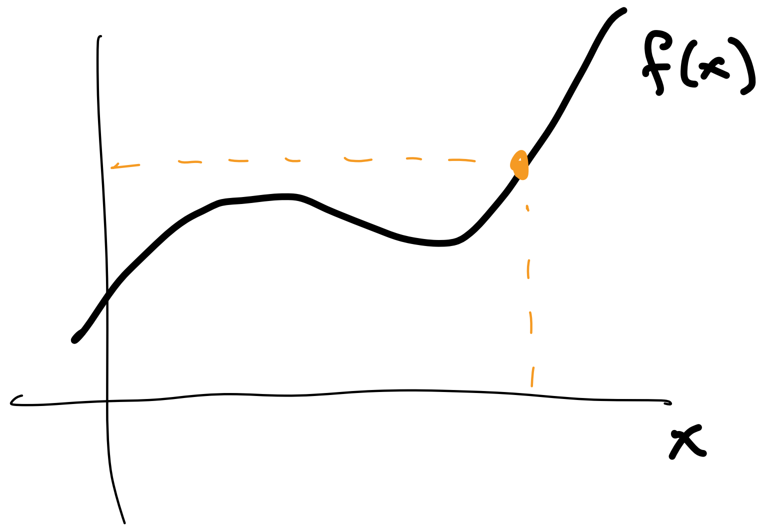
Continuity over p-adic numbers

23 Nov. 2021

AAG Seminar

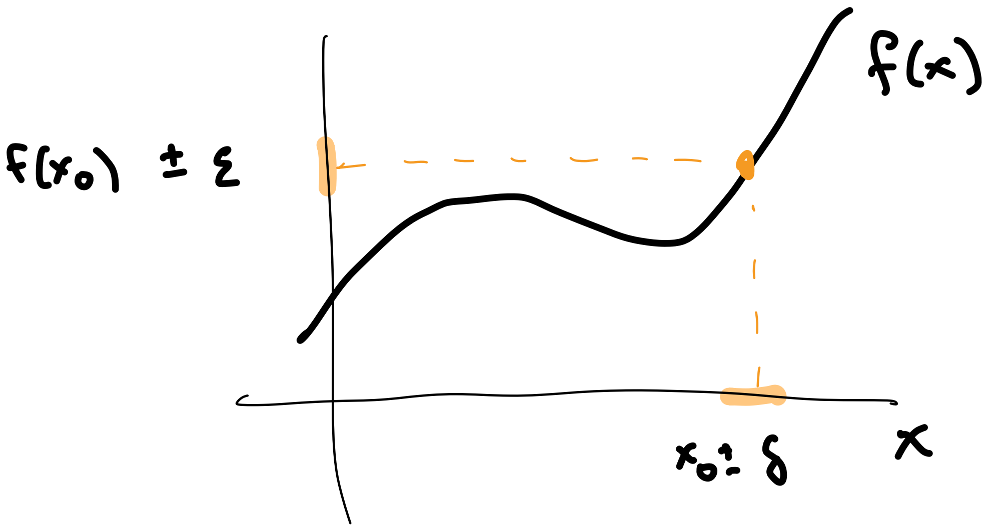
Continuity



Idea: small change in x $\rightsquigarrow$ small change in $f(x)$

Question: what is "small change"?

Continuity



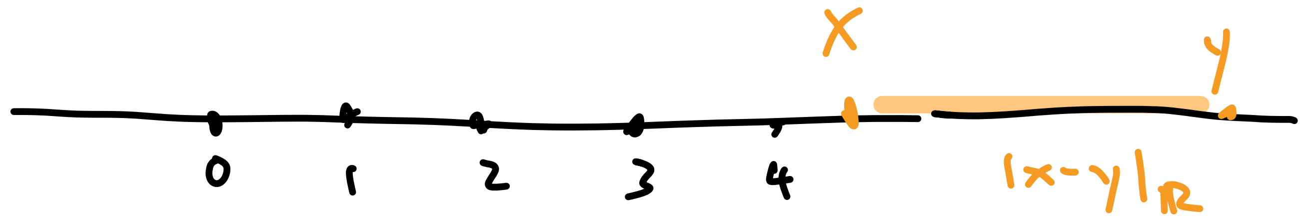
Idea: small change in x $\rightsquigarrow$ small change in $f(x)$

Def (ε-δ version) Input in range $(x_0-\delta, x_0+\delta)$ $\rightsquigarrow$ output in range $(f(x_0)-\varepsilon, f(x_0)+\varepsilon)$

$\forall \varepsilon \ \exists \delta$

Measuring distance

Usual distance:



$|x-y|_{\mathbb{R}}$ = "distance on number line"

$\varepsilon :=$ small constant $\Rightarrow$ $(-\varepsilon^n, \varepsilon^n)$ smaller range when n larger

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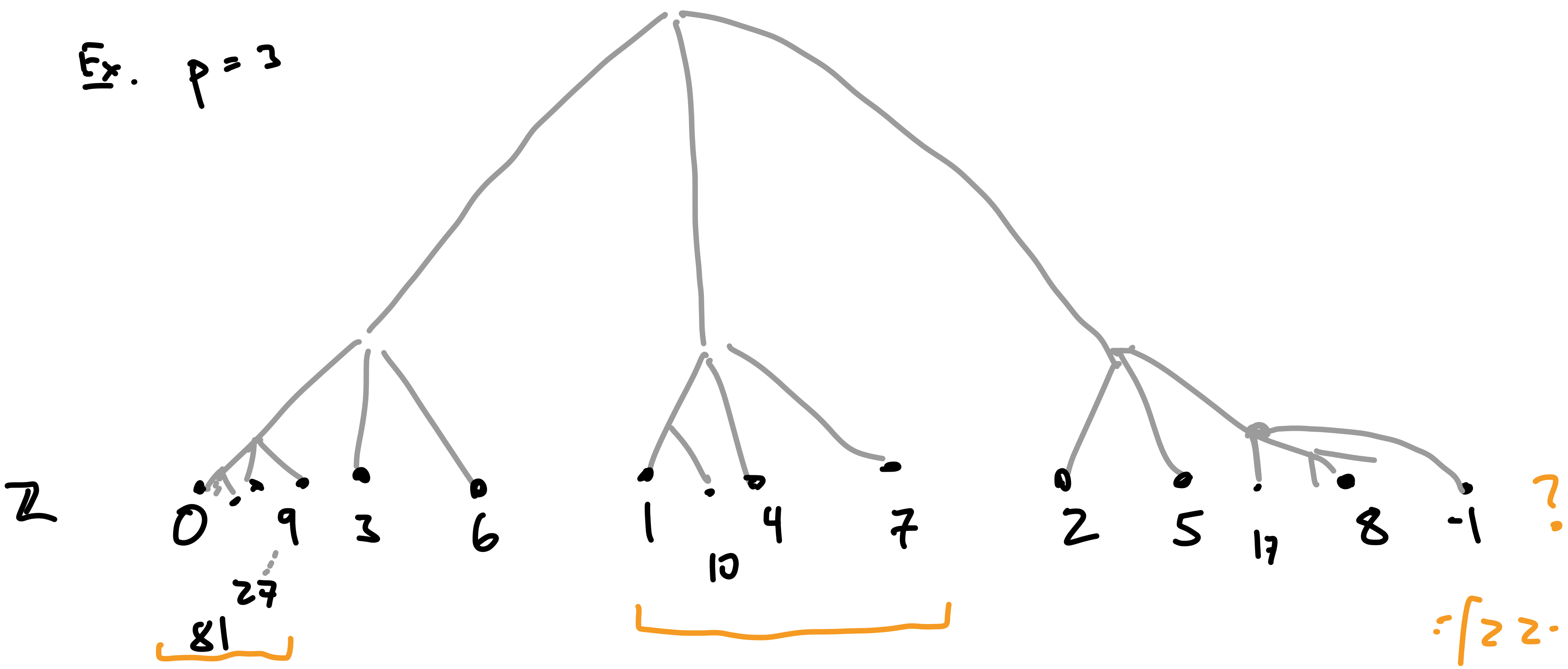
p-adic distance: p^n is "smaller" when n larger

small range around a = $(a - p^n, a + p^n)$ ✗

$= \{ a + kp^n : k \in \mathbb{Z} \}$

p-adic distance

Ex. $p = 3$



$\cdots(22\cdots)_3$

$1 + k \cdot 3$

$\Rightarrow 0 + k \cdot 27$

$81 + k \cdot 27$

$-27 + k \cdot 27$

p-adic continuity

Idea small change in x ⇝ small change in $f(x)$

Def $f(x)$ is p-adic continuous at x_0

if $\forall n, \exists m$ such that

$$x \in \{x_0 + kp^m\} \Rightarrow f(x) \in \{f(x_0) + kp^n\}$$

p-adic Continuity

Question: Is $f(x) = 10^x$ p-adic continuous?

Ex. $p = 2$, 2-adic continuity: $\Rightarrow$ not continuous at 0

$\Rightarrow$ continuous elsewhere?

x	10^x
0	1
1	10
2	100
4	10,000
8	10...0
$\vdots$	
2^n	10......0

close in 2-adic world

far 2-adically

close to 0 2-adically

p-adic Continuity

Question: Is $f(x) = 10^x$ p-adic continuous?

Ex. $p = 3$, 3-adic continuity: $\Rightarrow$ continuous (at 0) 3-adically

x	10^x
0	1
1	10
3	10^3
9	10^9
27	10^{27}
3^n	10^{3^n}

close 3-adically

close 3-adically

$3^? \mid (10^{3^n} - 1)$

$= (1+9)^{3^n} = 1 + 3^n(9) + \ldots$

p-adic Continuity

Question: Is $x^2 + 7x + 5$ p-adic continuous?

n	$f(n)$

Yes, · finite addition is p-adic continuous

· finite mult. —"—

Question: How can we decide p-adic continuity in general?

Mahler expansion

$\mathbb{R}$ - continuous functions often have Taylor expansion

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots$$

Def. The Mahler expansion of $f(x)$ is

$$f(x) = c_0 + c_1 \binom{x}{1} + c_2 \binom{x}{2} + \cdots = \sum c_k \binom{x}{k}$$

where

$$\binom{x}{k} := \frac{1}{k!} \underbrace{x(x-1)(x-2)\cdots(x-k+1)}_{\text{"falling factorial"} \quad x^{\underline{k}}}$$

Mahler expansion

Def. The Mahler expansion of $f(x)$ is

$$f(x) = c_0 + c_1\binom{x}{1} + c_2\binom{x}{2} + \cdots = \sum c_k \binom{x}{k}$$

Ex. $3x^2 + 5x + 1 = 1 + 8\binom{x}{1} + 6\binom{x}{2} + 0\binom{x}{3} + \cdots$

$$3^x = (1+2)^x = \underbrace{1 + 2\binom{x}{1} + 2^2\binom{x}{2} + 2^3\binom{x}{3} + \cdots}$$

converges when $x \in \mathbb{N}$

non neg. integer

Mahler expansion

Def. The Mahler expansion of $f(x)$ is

$$f(x) = c_0 + c_1\binom{x}{1} + c_2\binom{x}{2} + \cdots = \sum c_k \binom{x}{k}$$

Thm (Mahler 1958)

$f(x)$ is p-adic continuous $: \mathbb{N} \to \mathbb{Z}_p$ $\iff$ $|c_k|_p \to 0$ as $k \to \infty$

$\mathbb{R}$-analogue: smoothness / regularity $\iff$ Fourier series decay

Mahler expansion

Thm (Mahler 1958)

$f(x)$ is p-adic continuous $\iff$ $|c_k|_p \to 0$ as $k \to \infty$

Pf sketch ($\Leftarrow$) Suppose $|c_k|_p \to 0$ as $k \to \infty$,

Let $f_N(x) = \sum_{k=0}^{N} c_k \binom{x}{k}$ truncated series

For $x \in \mathbb{Z}_p$,

$$|f(x) - f_N(x)|_p = \left| \sum_{k=N+1}^{\infty} c_k \binom{x}{k} \right|_p$$

$$\le \max_{k \ge N+1} |c_k|_p \cdot \left|\binom{x}{k}\right|_p$$

Mahler expansion

Q: Is 10^x p-adic continuous ?

Expand: $10^x = (1+9)^x$

$$= 1 + 9\binom{x}{1} + 9^2\binom{x}{2} + \cdots$$

$\Rightarrow$ $c_k = 9^k$ Mahler coeff.

Upshot 10^x is p-adic continuous $\iff |9^k|_p \to 0$

$\iff |9|_p < 1, \quad p = 3$

How to find Mahler coefficients?

$$f(x) = \sum_{k\geq 0} c_k \binom{x}{k} \Rightarrow f(0) = c_0 + 0 + 0 + \dots$$

Def The finite difference operator

$$\Delta f(x) = f(x+1) - f(x)$$

Fact: $\Delta \binom{x}{k} = \binom{x+1}{k} - \binom{x}{k} = \binom{x}{k-1}$ $\qquad \frac{d}{dx} \frac{x^k}{k!} = \frac{x^{k-1}}{(k-1)!}$

$$\Rightarrow \Delta\left(\sum_{k\geq 0} c_k \binom{x}{k}\right) = \sum_{k\geq 1} c_k \binom{x}{k-1}$$

$$\Delta f(0) = c_1, \quad \Delta^2 f(0) = c_2, \quad \dots, \quad \Delta^k f(0) = c_k .$$

Mahler expansion

Q: Is $n!$ p-adic continuous?

$\Rightarrow$ What is its Mahler expansion?

n	$n!$
0	1
1	1
2	2
3	6
4	24
5	120