Descartes' Rule of Signs and Beyond

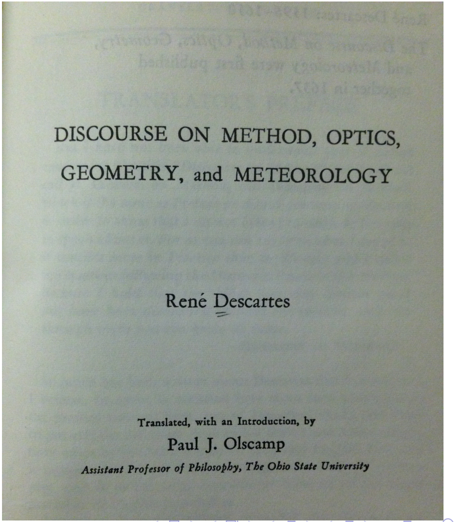
Harry Richman

University of Michigan

September 21, 2017

René Descartes

- 1596 - 1650
- French philosopher, scientist, mathematician



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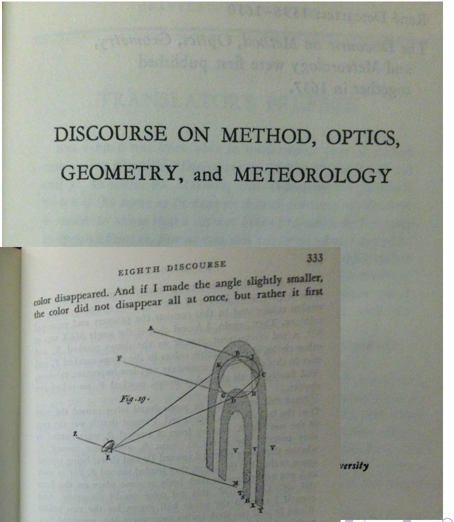
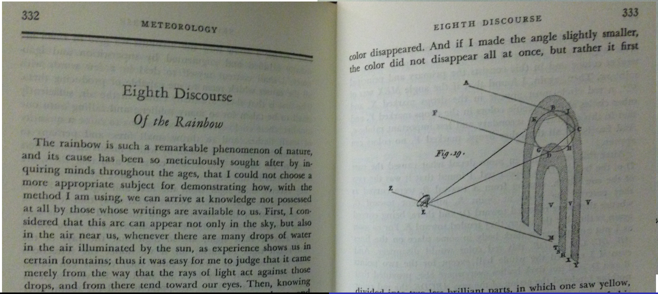

DISCOURSE ON METHOD, OPTICS, GEOMETRY, and METEOROLOGY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Eighth Discourse

Of the Rainbow

The rainbow is such a remarkable phenomenon of nature, and its cause has been so meticulously sought after by inquiring minds throughout the ages, that I could not choose a more appropriate subject for demonstrating how, with the method I am using, we can arrive at knowledge not possessed at all by those whose writings are available to us. First, I considered that this arc can appear not only in the sky, but also in the air near us, whenever there are many drops of water in the air illuminated by the sun, as experience shows us in certain fountains; thus it was easy for me to judge that it came merely from the way that the rays of light act against those drops, and from there tend toward our eyes. Then, knowing

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color disappeared. And if I made the angle slightly smaller, the color did not disappear all at once, but rather it first

Fig. 19.

divided into two less brilliant parts, in which one saw yellow,

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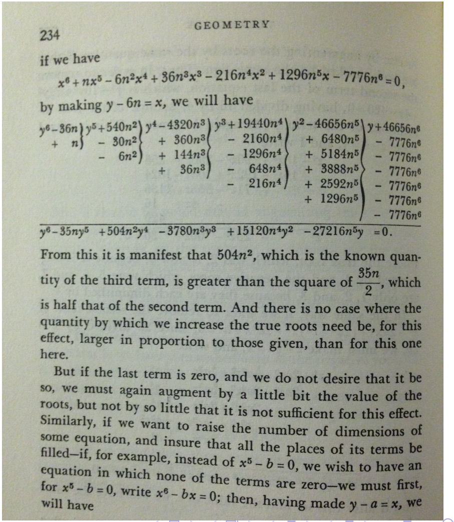
234 GEOMETRY

if we have

$$x^6 + nx^5 - 6n^2x^4 + 36n^3x^3 - 216n^4x^2 + 1296n^5x - 7776n^6 = 0,$$

by making $y - 6n = x$, we will have

$$\begin{array}{llllll}
y^6 - 36n & y^5 + 540n^2 & y^4 - 4320n^3 & y^3 + 19440n^4 & y^2 - 46656n^5 & y + 46656n^6 \\
\quad + n & \quad - 30n^2 & \quad + 360n^3 & \quad - 2160n^4 & \quad + 6480n^5 & \quad - 7776n^6 \\
 & \quad - 6n^2 & \quad + 144n^3 & \quad - 1296n^4 & \quad + 5184n^5 & \quad - 7776n^6 \\
 & & \quad + 36n^3 & \quad - 648n^4 & \quad + 3888n^5 & \quad - 7776n^6 \\
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\hline
y^6 - 35ny^5 & + 504n^2y^4 & - 3780n^3y^3 & + 15120n^4y^2 & - 27216n^5y & = 0.
\end{array}$$

From this it is manifest that $504n^2$, which is the known quantity of the third term, is greater than the square of $\frac{35n}{2}$, which is half that of the second term. And there is no case where the quantity by which we increase the true roots need be, for this effect, larger in proportion to those given, than for this one here.

But if the last term is zero, and we do not desire that it be so, we must again augment by a little bit the value of the roots, but not by so little that it is not sufficient for this effect. Similarly, if we want to raise the number of dimensions of some equation, and insure that all the places of its terms be filled—if, for example, instead of $x^5 - b = 0$, we wish to have an equation in which none of the terms are zero—we must first, for $x^5 - b = 0$, write $x^6 - bx = 0$; then, having made $y - a = x$, we will have

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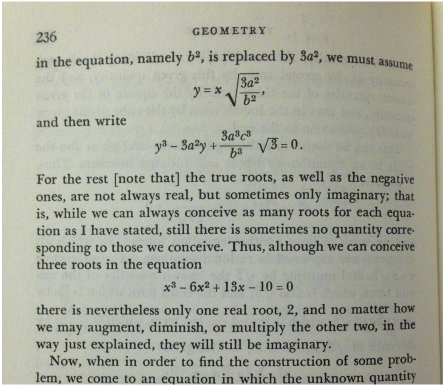

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in the equation, namely b^2, is replaced by $3a^2$, we must assume

$$y = x\sqrt{\frac{3a^2}{b^2}},$$

and then write

$$y^3 - 3a^2y + \frac{3a^3c^3}{b^3}\sqrt{3} = 0.$$

For the rest [note that] the true roots, as well as the negative ones, are not always real, but sometimes only imaginary; that is, while we can always conceive as many roots for each equation as I have stated, still there is sometimes no quantity corresponding to those we conceive. Thus, although we can conceive three roots in the equation

$$x^3 - 6x^2 + 13x - 10 = 0$$

there is nevertheless only one real root, 2, and no matter how we may augment, diminish, or multiply the other two, in the way just explained, they will still be imaginary.

Now, when in order to find the construction of some problem, we come to an equation in which the unknown quantity

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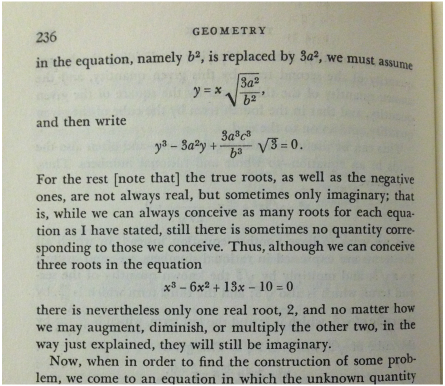

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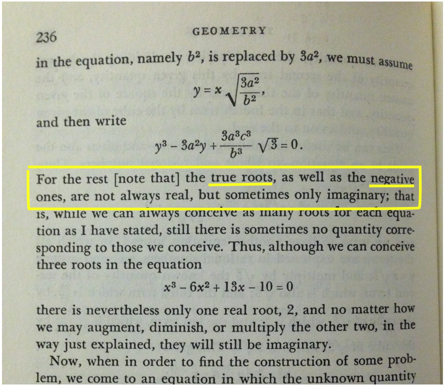

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Polynomials: example

$$f(x) = x^2 - 8x + 2$$

- Which $x > 0$ satisfy $f(x) = 0$?
- How many $x > 0$ satisfy $f(x) = 0$?

Polynomials: example

$$f(x) = x^2 - 8x + 2 \qquad \text{(degree=2)}$$

- Which $x > 0$ satisfy $f(x) = 0$?

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- How many $x > 0$ satisfy $f(x) = 0$? evaluate formula $\uparrow$

Polynomials: example

$$f(x) = x^{10} + 7x^2 - 8x + 2 \qquad \text{(degree=10)}$$

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A clever shortcut!

- How many x satisfy $f(x) = 0$?

We can also know from this how many true and how many negative roots there can be in each equation, namely, there can be as many true roots as the number of times the plus and minus signs change; and as many negative roots as the number of times there are two plus or two minus signs in succession. Thus, in the last equation, since $+x^4$ is followed by $-4x^3$, which is a change from the plus sign to the minus, and $-19x^2$ is followed by $+106x$, and $+106x$ by -120, which are two more changes, we know that there are three true roots; and because

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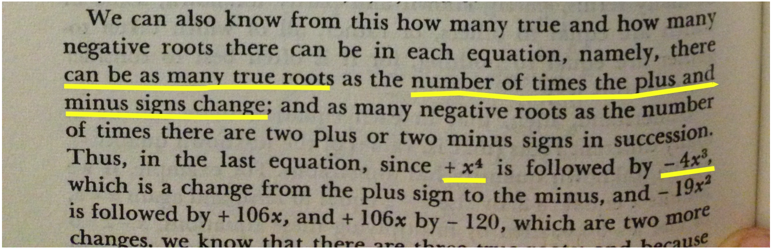

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For a polynomial with real coefficients,

$$\#(\textit{positive real roots}) \leq \#(\textit{sign changes of coefficients}).$$

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2 sign changes $\Rightarrow \leq 2$ real pos. roots

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Challenge

Prove this for $f(x) = ax^2 \pm bx \pm c$.

Why guess this?

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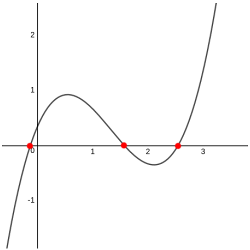
- $f(x) = 0$ means graph changes $(+-)$ or $(-+)$



Why guess this? a "'"fake" "proof"'

- plot points $f(x) = 1 + 7x - 8x^2 + 2x^3$

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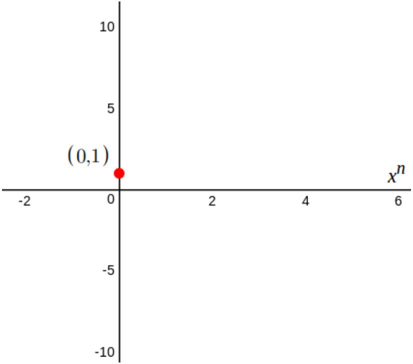

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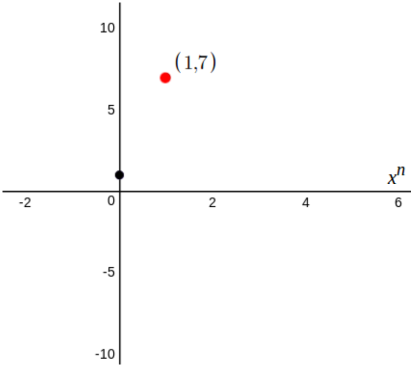

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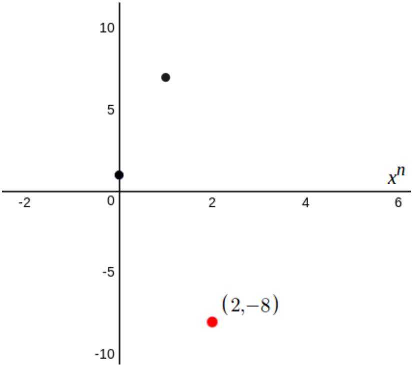

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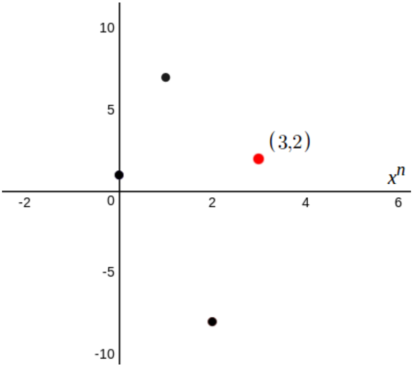

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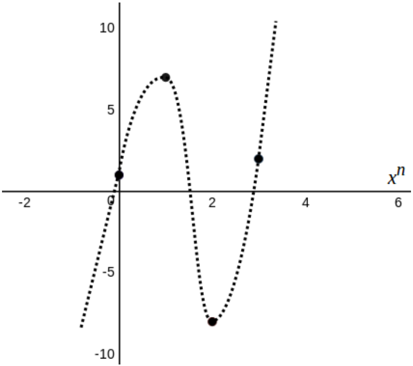

- count roots!

Why guess this? a “‘‘fake” “proof” ’

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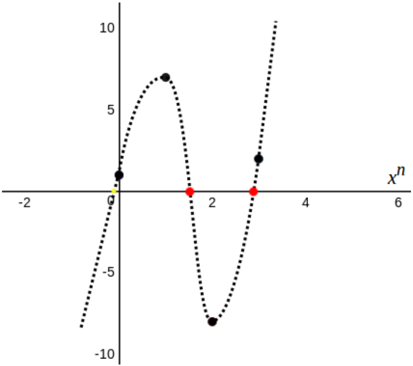

- count roots! (????)

Is this justified???

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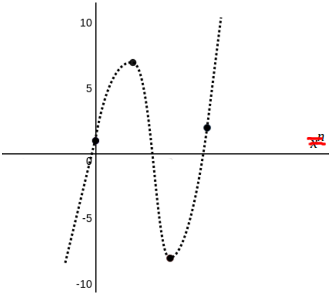

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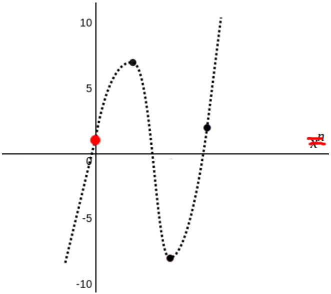

- when $x = 0$, $\qquad f(0) = 1 + 0 + 0 + 0 = 1 > 0$

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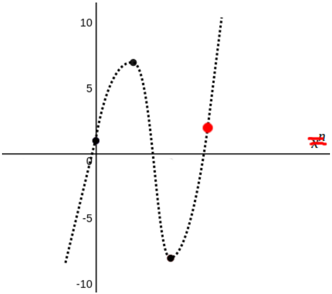

- when
  $x = N \gg 0, \qquad f(N) = 1 + 7N - 8N^2 + 2N^3 \approx 2N^3 > 0$

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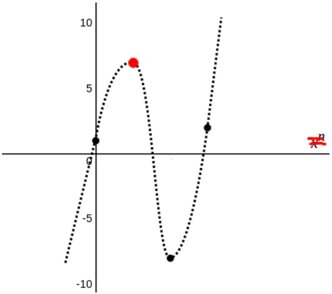

- when $x = ??$, $\qquad f(x = ??) \approx 7x^1 > 0$

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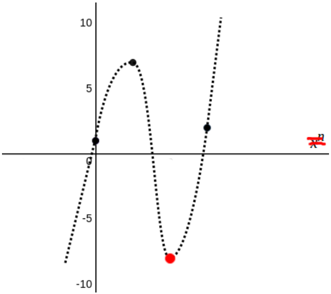

- when $x =??,\qquad f(x =??) \approx -8x^2 < 0$

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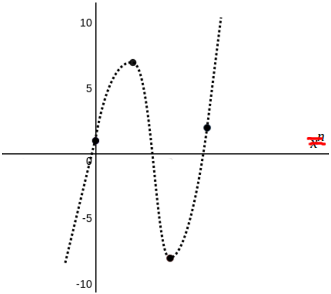

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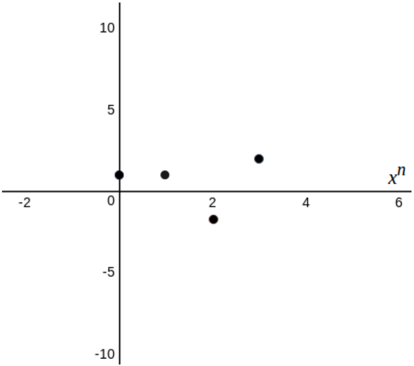

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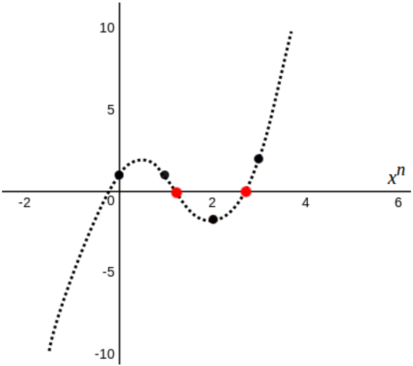

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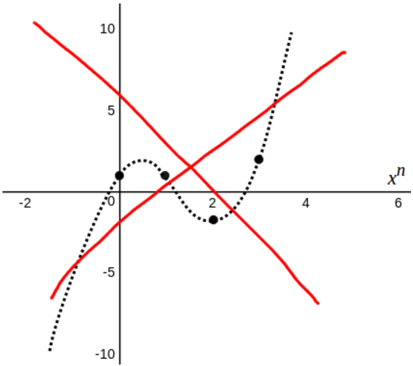

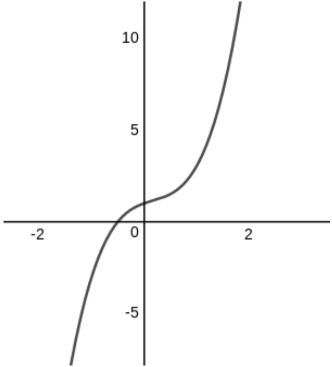

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Source: REI.com

Concavity

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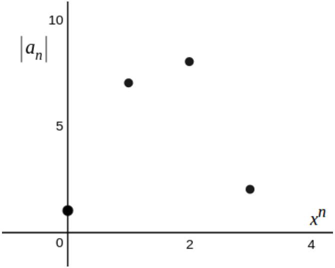
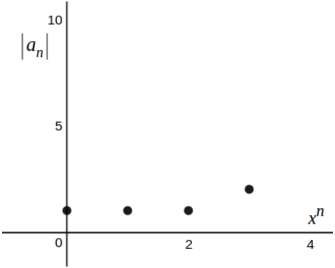


$\Rightarrow$ yes



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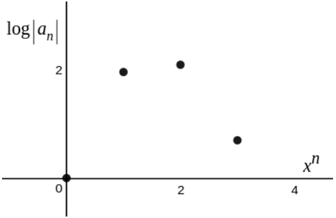

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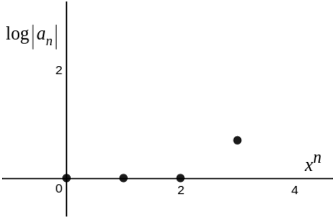

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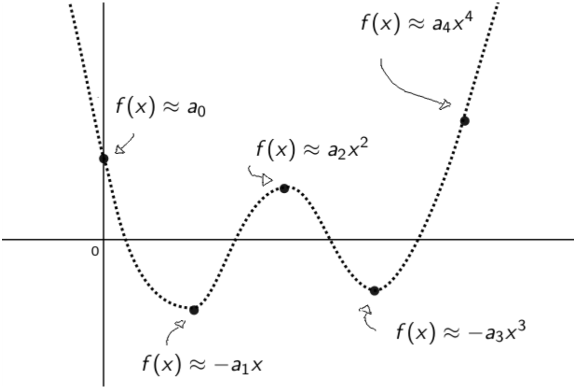

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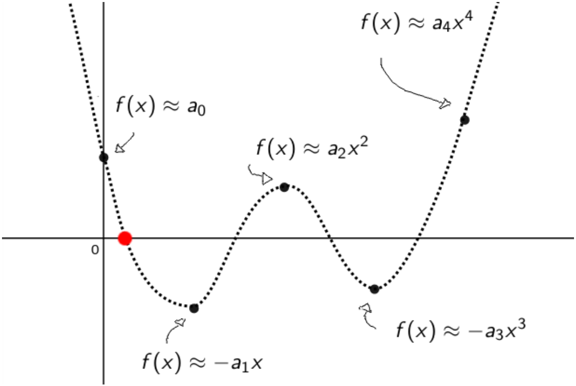

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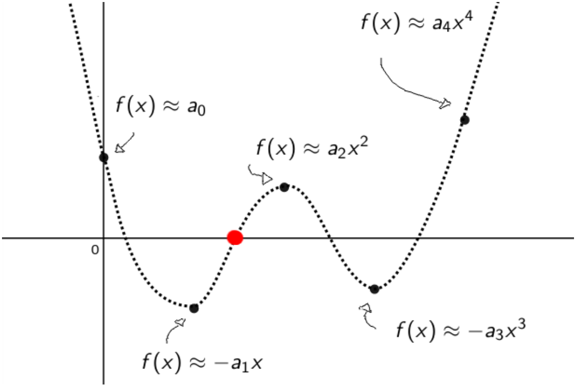

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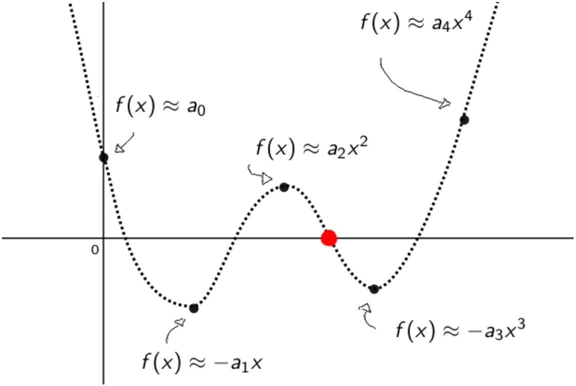

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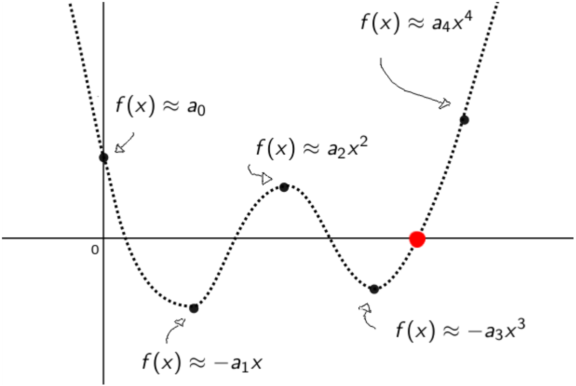

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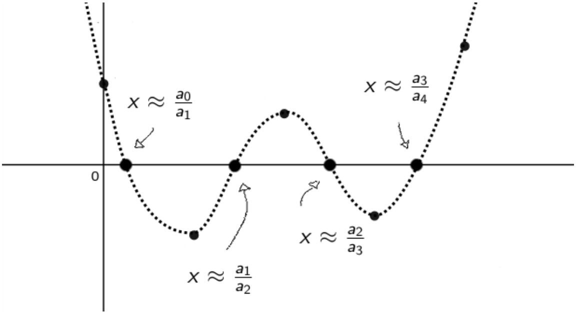

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log-Concavity

Theorem (Newton, via Stanley)

If

$$f(x) = a_0 - a_1 x + a_2 x^2 - \cdots \pm a_n x^n \qquad (a_i > 0)$$

has all real roots, then the sequence

$$a_0 / \binom{n}{0}, a_1 / \binom{n}{1}, \ldots, a_n / \binom{n}{n}$$

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- Challenge: prove this!

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$\Rightarrow$ "non-Archimedean" or "ultrametric" fields

Non-Archimedean fields: motivation

Problem

How big is $a + b$ compared to a, b?

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  i.e. ALL differences are very large

Non-Archimedean fields

- field K with valuation $val : K^\times \to \mathbb{R}$

Idea: *val* measures how "big" a number is,
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  2. if $val(a) \neq val(b)$,

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  3. In general,
$$val(a+b) \leq \max\{val(a), val(b)\}$$
  4. (also: $val(0) = -\infty$)

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$$K = \mathbb{R}(\epsilon) = \{a_n\epsilon^n + a_{n+1}\epsilon^{n+1} + \cdots\},$$

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  - p-adic numbers

$$K = \mathbb{Q},$$

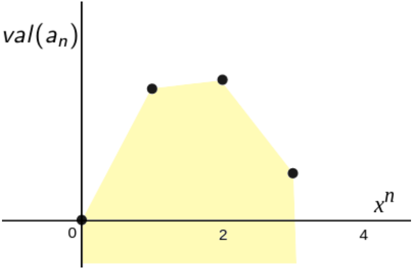
$$val : p^n \mapsto -n, \qquad r \mapsto 0$$

Newton polygon

Given polynomial with coefficients in $K = \mathbb{R}(\epsilon)$, e.g.

$$f(x) = (1 + 2\epsilon) + \epsilon^{-7}x + (\epsilon^{-8} + 3\epsilon^{-1} + 1 + \epsilon^{5})x^2 + \epsilon^{-2}x^3,$$

the **Newton polygon** is the lower-convex hull of the graph $val(a_n)$:



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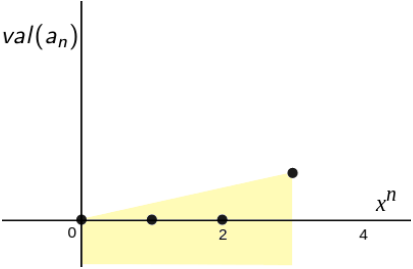


- using Newton polygon leads to better* rule of signs!

Newton polygon + Descartes' rule

- $K = \mathbb{R}(\epsilon, \epsilon^{1/2}, \epsilon^{1/3}, \ldots)$ rational power series* in ϵ
- a number is "positive" if its leading term is positive

Theorem (non-Archimedean Descartes' rule)

For $f(x) \in K[x]$, suppose that Newton polygon has "corners" at all points on boundary. Then

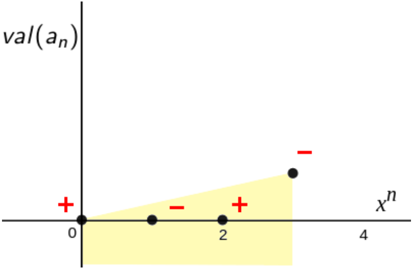
$$\#(\textit{positive real roots}) = \#(\textit{sign changes of Newton poly.}).$$

*really, need to take "completion" w.r.t. valuation

Newton polygon + Descartes' rule

- Example 1:

$$f(x) = +(1+2\epsilon) - (7-\epsilon^4)x + (3+\epsilon^5)x^2 - (\epsilon^{-2}+1)x^3$$



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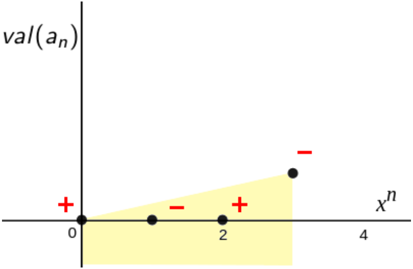

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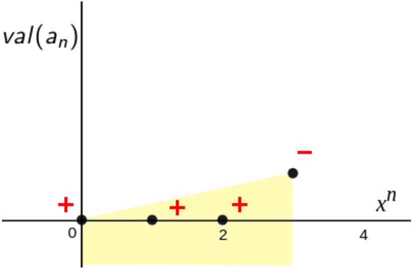

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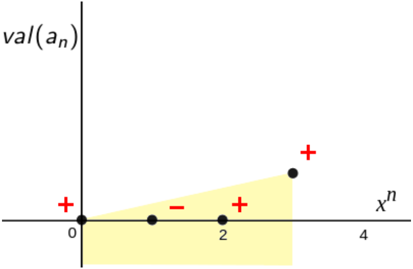
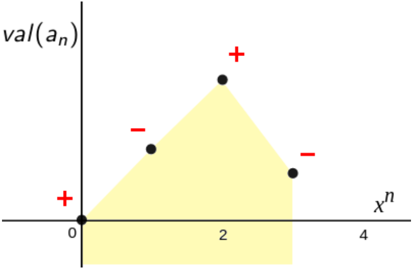


0 sign changes $\Rightarrow$ 0 pos. real roots

Newton polygon + Descartes' rule

- Example 2:

$$f(x) = +(1+2\epsilon) - \epsilon^{-4}x + (\epsilon^{-8} + 3\epsilon^{-1} + \epsilon^{5})x^2 - (\epsilon^{-2}+1)x^3$$



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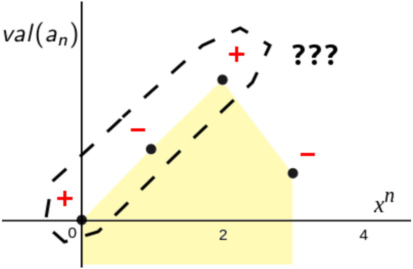

3 sign changes $\Rightarrow \leq 3$ pos. real roots (usual Descartes' rule)

Newton polygon + Descartes' rule

- Example 3:

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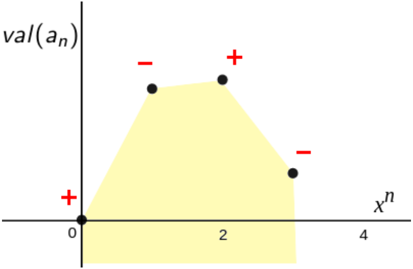

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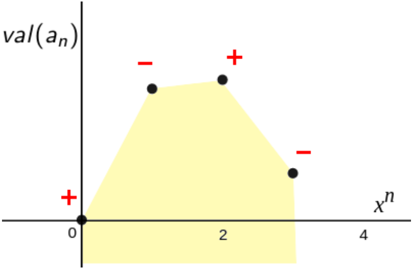

3 sign changes ⇒ 3 pos. real roots

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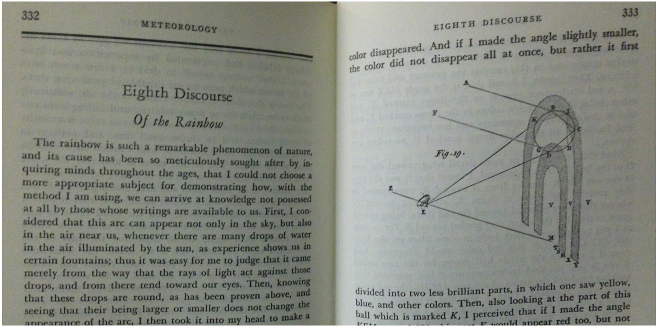
Beyond Descartes' rule



332 METEOROLOGY

Eighth Discourse

Of the Rainbow

The rainbow is such a remarkable phenomenon of nature, and its cause has been so meticulously sought after by inquiring minds throughout the ages, that I could not choose a more appropriate subject for demonstrating how, with the method I am using, we can arrive at knowledge not possessed at all by those whose writings are available to us. First, I considered that this arc can appear not only in the sky, but also in the air near us, whenever there are many drops of water in the air illuminated by the sun, as experience shows us in certain fountains; thus it was easy for me to judge that it came merely from the way that the rays of light act against those drops, and from there tend toward our eyes. Then, knowing that these drops are round, as has been proven above, and seeing that their being larger or smaller does not change the appearance of the arc, I then took it into my head to make a

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color disappeared. And if I made the angle slightly smaller, the color did not disappear all at once, but rather it first

Fig. 19.

divided into two less brilliant parts, in which one saw yellow, blue, and other colors. Then, also looking at the part of this ball which is marked *K*, I perceived that if I made the angle *KEM* ... would appear red too, but not

Thank you!