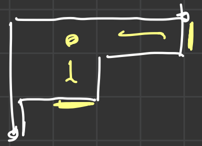
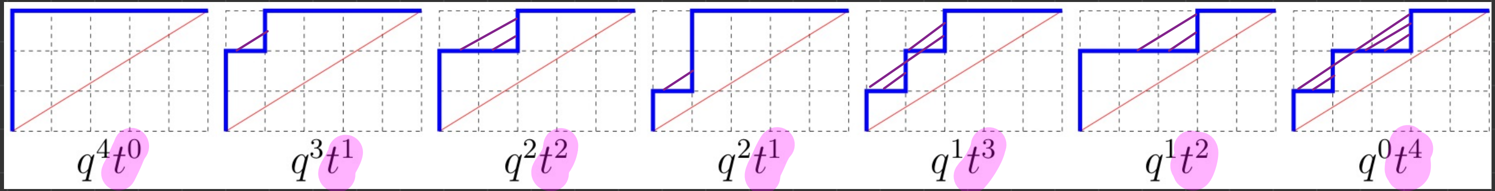
Dyck path statistic: dinv

Note: pairs (h, w) corresp. to boxes above Dyck path:

↗ right ↖ left



⇒ $\text{dinv}(D) =$ count subset of boxes above Dyck path



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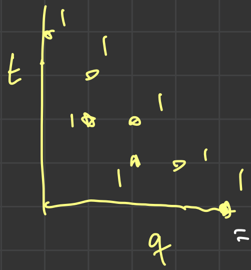
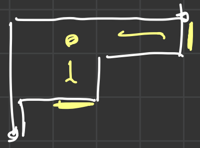
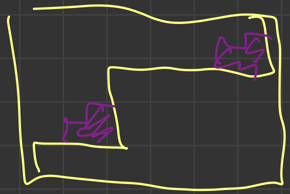
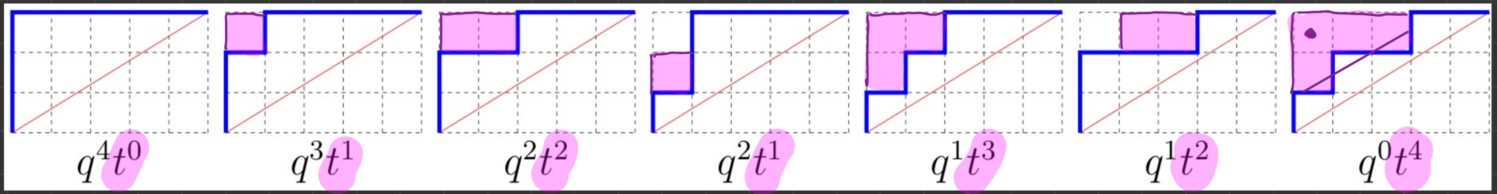




Q: can there be holes in corresp. boxes?



$\Rightarrow$ $\text{dinv}(D)$ = count subset of boxes above Dyck path



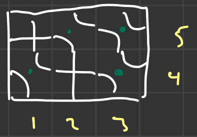
area + dinv: 4 4 4 3 4 3 4

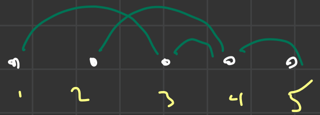
$C_{a,b}(q,q)$ = ? $\Big|$ $q^{\text{area}(D)}\, t^{\text{dinv}(D)}$ ⇝ $\text{area}(D) + \text{dinv}(D) \le \left(\begin{matrix}\#\text{ boxes}\\ \text{above diag.}\end{matrix}\right) = \dfrac{(a-1)(b-1)}{2}$

Day 2 discussion

- Suggestion: classify all (3,4) diagrams

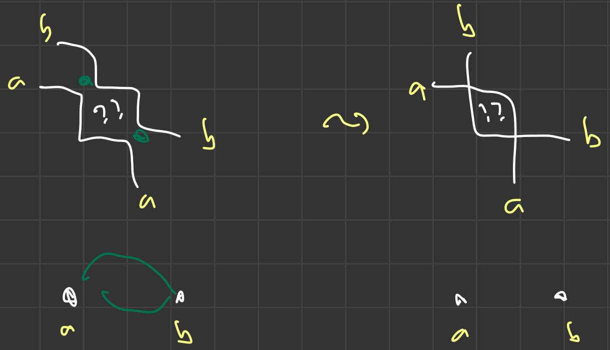
- From maximal diagram, construct the "wire-elbow" graph





Q: Is this graph always a tree? It has $\# V = a + b$ and $\# E = a + b - 1$, so suffices to check [connected] <=> [no cycles]

$\exists$ 2-cycles?



- Consider "platform levels" of diagram, erasing covered wires

- consider wires on torus

Numerical semigroups

A numerical semigroup (Γ) is a subset $S \subset \mathbb{N} = \{0, 1, 2, \dots\}$

which satisfies

- $0 \in S$
- (semigroup) closed under addition → $\mathbb{N} \setminus S$ = "gaps" of S
- cofinite $\#(\mathbb{N} \setminus S) < \infty$

abacus diagram

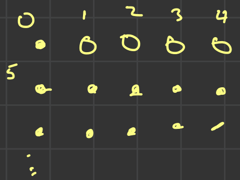
Ex. $\langle 3, 5 \rangle = 3\mathbb{N} + 5\mathbb{N} = \{0, 3, 6, \dots\} + \{0, 5, 10, \dots\}$

$= \{0, 3, 5, 6, 8, 9, 10, \dots\}$

$= \mathbb{N} \setminus \{1, 2, 4, 7\}$

Ex. $\langle 5, 6, 7, 8, 9 \rangle = 5\mathbb{N} + \dots + 9\mathbb{N} = \{0, 5, 6, 7, \dots\}$

$= \mathbb{N} \setminus \{1, 2, 3, 4\}$



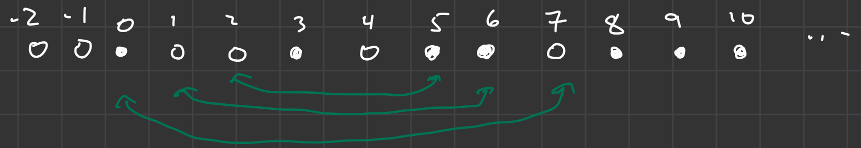
Numerical semigroups

A two-generator semigroup $S = \langle a, b \rangle$ for a, b coprime has the following symmetry property:

$$n \in S \iff \underbrace{ab - a - b}_{} - n \notin S$$

$ab - a - b$ = largest gap = "Frobenius number"

Ex. $S = \langle 3, 5 \rangle =$



$n \mapsto 7 - n$ $\qquad 7 = 3\cdot 5 - 3 - 5$

Generally, a symmetric numerical semigroup satisfies

$$n \in S \iff c - n \notin S \quad \text{for some } c = c(S)$$

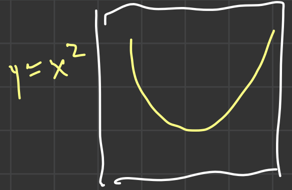
$$(\text{2-generator}) \subsetneq (\text{symmetric}) \subsetneq (\text{all num. semigroups})$$

Algebraic curves

[Goal: geometric interpretation & dim]

$\mathbb{C}[t]$ = regular functions on affine line $\mathbb{A}^1$

"Same" curve embedded in plane $\mathbb{A}^2$:



$$C = \operatorname{Spec}\left(\mathbb{C}[x,y] / (y - x^2) \right) \xleftarrow{\cong} \operatorname{Spec} \mathbb{C}[t] = \mathbb{A}^1$$

$$\begin{array}{ccc} \mathbb{C}[x,y] & \xrightarrow{\varphi} & \mathbb{C}[t] \\ x & \mapsto & t \\ y & \mapsto & t^2 \end{array}$$

$$\ker(\varphi) = (\overset{t^2 - t^2}{y - x^2})$$

$$\operatorname{im}(\varphi) = \mathbb{C}[t]$$

$\Rightarrow$ φ induces isomorphisms

$$\bar{\varphi} : \frac{\mathbb{C}[x,y]}{(y - x^2)} \xrightarrow{\sim} \mathbb{C}[t] \quad \Leftrightarrow \quad \bar{\varphi}^* : \mathbb{A}^1 \xrightarrow{\sim} C$$

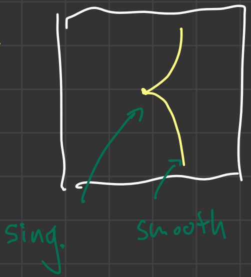
reg. functions

points

Algebraic curves

Singular curve embedded in plane $\mathbb{A}^2$:

$y^2 = x^5$



$$C = \operatorname{Spec}\left(\mathbb{C}[x,y]/(y^2-x^5)\right) \xleftarrow{\varphi^*} \operatorname{Spec}\mathbb{C}[t] = \mathbb{A}^1$$

$t^{10} - t^{10} = 0$

$\mathbb{C}[x,y] \xrightarrow{\varphi} \mathbb{C}[t]$, $\ker(\varphi) = (y^2 - x^5)$

$x \mapsto t^2$

$y \mapsto t^5$

$\operatorname{im}(\varphi) = \mathbb{C}[t^2, t^5] \not\ni t$

↳ not surjective! t^3

$\uparrow$ normalization, resolves singularity

$\mathbb{A}^1$

$\Rightarrow$ φ induces morphisms

$$\bar{\varphi}: \frac{\mathbb{C}[x,y]}{(y^2-x^5)} \xrightarrow{\sim} \mathbb{C}[t^2,t^5] \subsetneq \mathbb{C}[t]$$

$$C \xleftarrow{\sim} \operatorname{Spec}(\mathbb{C}[t^2,t^5]) \twoheadleftarrow \mathbb{A}^1$$

Note: normalization map here is bijective on points, but not isomorphism of curves since regular functions are non-isomorphic rings

Algebraic curves

Given numerical semigroup $S \subset \mathbb{N}$, construct singular curve

$$C = C_S = \operatorname{Spec}(\mathbb{C}[t^n : n \in S])$$

$\cap$ subring

$\mathbb{C}[t]$

Ex $S = \langle 2, 5 \rangle$, C_S embeds in $\mathbb{A}^2$

Ex. $S = \langle 5, 6, 7, 8 \rangle$, C_S embeds in $\mathbb{A}^4$

Semigroup modules

A module for a semigroup $S \subset \mathbb{N}$ is a subset $M \subset \mathbb{N}$ such that
(semi-)

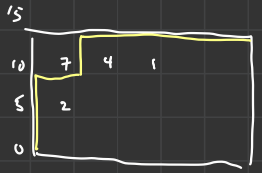
$$S + M \subset M$$

A module is 0-normalized if $0 \in M$ $\left(0 \in M \text{ and } S + M \subset M \implies S \subset M \right)$

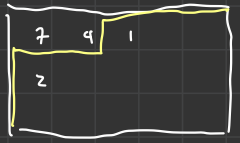
Fact: A numerical semigroup S has fin.-many 0-norm. modules.

Ex. $S = \langle 3,5 \rangle = \{0, \overset{1}{\ } \overset{2}{\ } 3, \overset{4}{\ } 5, 6, \overset{7}{\ } 8, 9, \dots\}$

$M_1 = \{0, 3, 5, 6, 7, 8, 9, \dots\}$ →



$M_2 = \{0, 3, 4, 5, 6, 7, 8, 9, \dots\}$ →



Semigroup modules

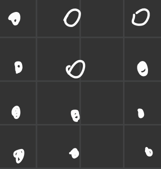
Let M = 0-normalized module for 2-generator semigroup $S = \langle a, b \rangle$

The *a-basis* of M is the set corresp. to top bead in each column of a-abacus diagram

Ex. $M =$

3-basis $= \{0, 10, 5\} = \{0, 5, 10\}$ increasing order

Ex. $M =$



3-basis $= \{0, 7, 5\} = \{0, 5, 7\}$

Semigroup modules & Jacobians

Fact: 0-normalized modules of $S=\langle a,b\rangle$ are in bijection with Dyck paths $\mathcal{D}_{a,b}$

Theorem (Gorsky-Mazin Thm 2.8) The singular plane curve C_S, $S=\langle a,b\rangle$ has compactified Jacobian $\overline{JC}_S$ which decomposes into affine cells

$$\overline{JC}_S = \bigcup_M C_M$$

indexed by 0-norm. modules of S. Each cell has dimension

$$\dim C_M = \frac{1}{2}(a-1)(b-1) - \operatorname{dinv}(\underset{\uparrow}{D}(M))$$

corresp. Dyck path