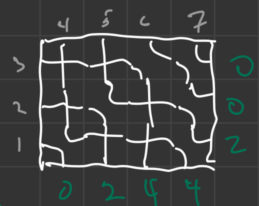
Day 1 discussion

- Given maximal diagram, count # crossings on each wire in order from 1, 2, ..., a+b to get a "crossing sequence"

#crossings:



$\rightsquigarrow$ crossing sequence $(0, 2, 4, 4, 2, 0, 0)$

Q: Is crossing sequence always unimodal on max. diagrams?

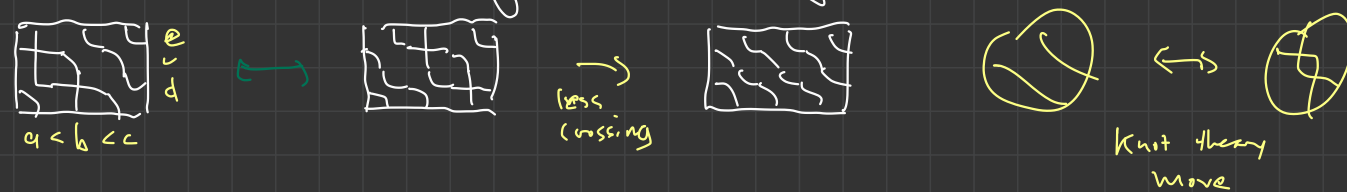
- In a maximal diagram, can two columns (resp. rows) have same pattern & crossings vs. elbows?

Small examples suggest distinct columns have distinct crossing/elbow patterns.

Day 1 discussion

- Which 'local moves' send deograms to deograms?

e.g.



- For coprime (a,b) we want bijection realizing

$$\#\{(a,b)\text{ deograms}\} = \#\{(a,b)\text{-Dyck paths}\} = \frac{1}{a+b}\binom{a+b}{a}$$

Is it true that for non-coprime (a,b)

$$\#\{(a,b)\text{ deograms}\} = \#\{(a,b)\text{-Dyck paths}\} \neq \frac{1}{a+b}\binom{a+b}{a}$$

Answer: better to ignore non-coprime case for now

q, t - rational Catalan polynomials ("q, t ... numbers")

Yesterday: coprime (a,b) $\rightsquigarrow$ integer $C_{a,b} = \#\{a,b \text{ Dyck paths}\}$

$= \# \mathcal{D}_{a,b}$

Refinement: coprime (a,b) $\rightsquigarrow$ polynomial

$$C_{a,b}(q,t) = \sum_{D \in \mathcal{D}_{a,b}} q^{\operatorname{area}(D)} t^{\operatorname{dinv}(D)}$$

using statistics "area" and "dinv" on Dyck paths

[Note that
$$C_{a,b}(1,1) = C_{a,b}$$
]

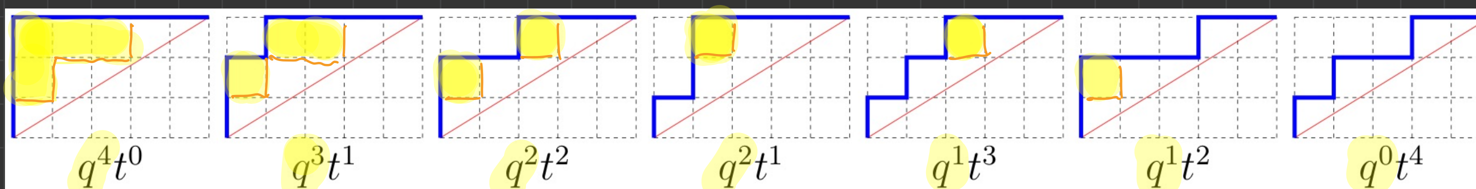
Historically, first considered

$$C_{a,b}(q) = \sum_{D \in \mathcal{D}_{a,b}} q^{\operatorname{area}(D)}$$

Dyck path statistic : area

area(D) = # boxes between Dyck path and (a,b)-diagonal

$$C_{a,b}(q,t) = \sum_{D \in \mathcal{D}_{a,b}} q^{\text{area}(D)}\, t^{\text{dinv}(D)}$$



Note: area(D) = size of 1st column in corresp. (a,b)-core

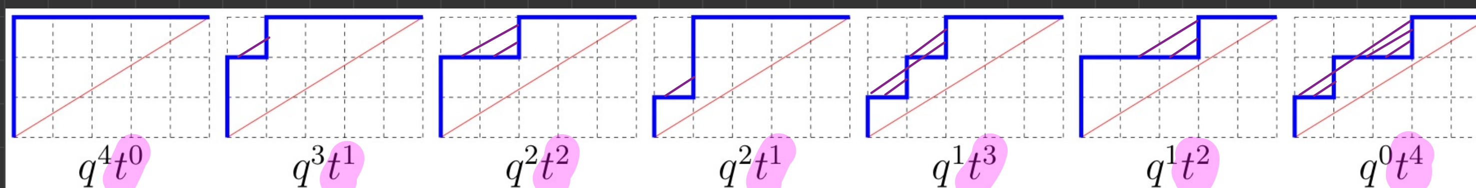
= # rows in —— " ——

Dyck path statistic : dinv

$\mathrm{dinv}(D) = \#$ pairs (h, v) where h = horiz. step (left-ward), v = vertical step (right-ward)

such that slope - a/b segment connects h and v

slope of main diagonal



$\{\mathrm{dinv}\}_{5,1} = \{0, 1, 1, 2, 2, 3, 4\}$

$$C_{a,b}(q,t) = \sum_{D \in \mathcal{D}_{a,b}} q^{\mathrm{area}(D)} t^{\mathrm{dinv}(D)}$$

Thm $C_{a,b}(q,t) = C_{a,b}(t,q)$

Dyck path statistic: $\mathrm{dinv}(D) = \mathrm{area}(\mathrm{sweep}(D))$ → "modified area statistic"

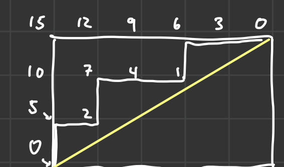
Thm

sweep: $\mathcal{D}_{a,b} \longrightarrow \mathcal{D}_{a,b}$ defines bijection on Dyck paths as follows:

- label lattice pt (i,j) by number $bi - aj$
- label each step in D by starting (ending) vertex
- sort steps by increasing (decreasing) label

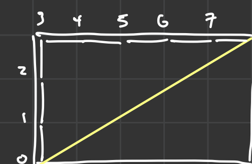
Ex.

$bi - aj$



N E N E E N E E
0 5 2 7 4 1 6 3

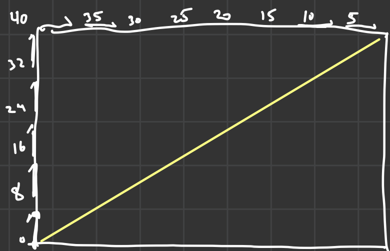
$\xrightarrow{\text{sweep}}$



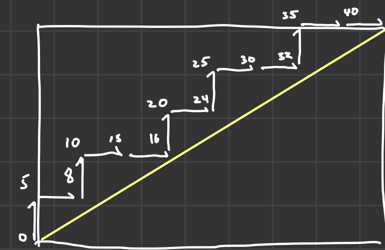
N N N E E E E E
0 1 2 3 4 5 6 7

Dyck paths : sweep map

Ex. C(8,5)

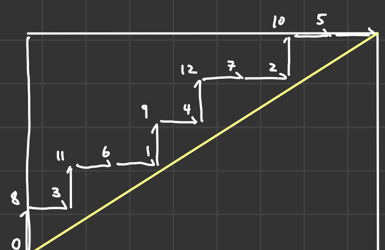


sweep
⇝

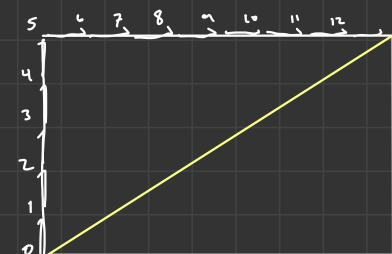


N N N N N E E E E E E E E

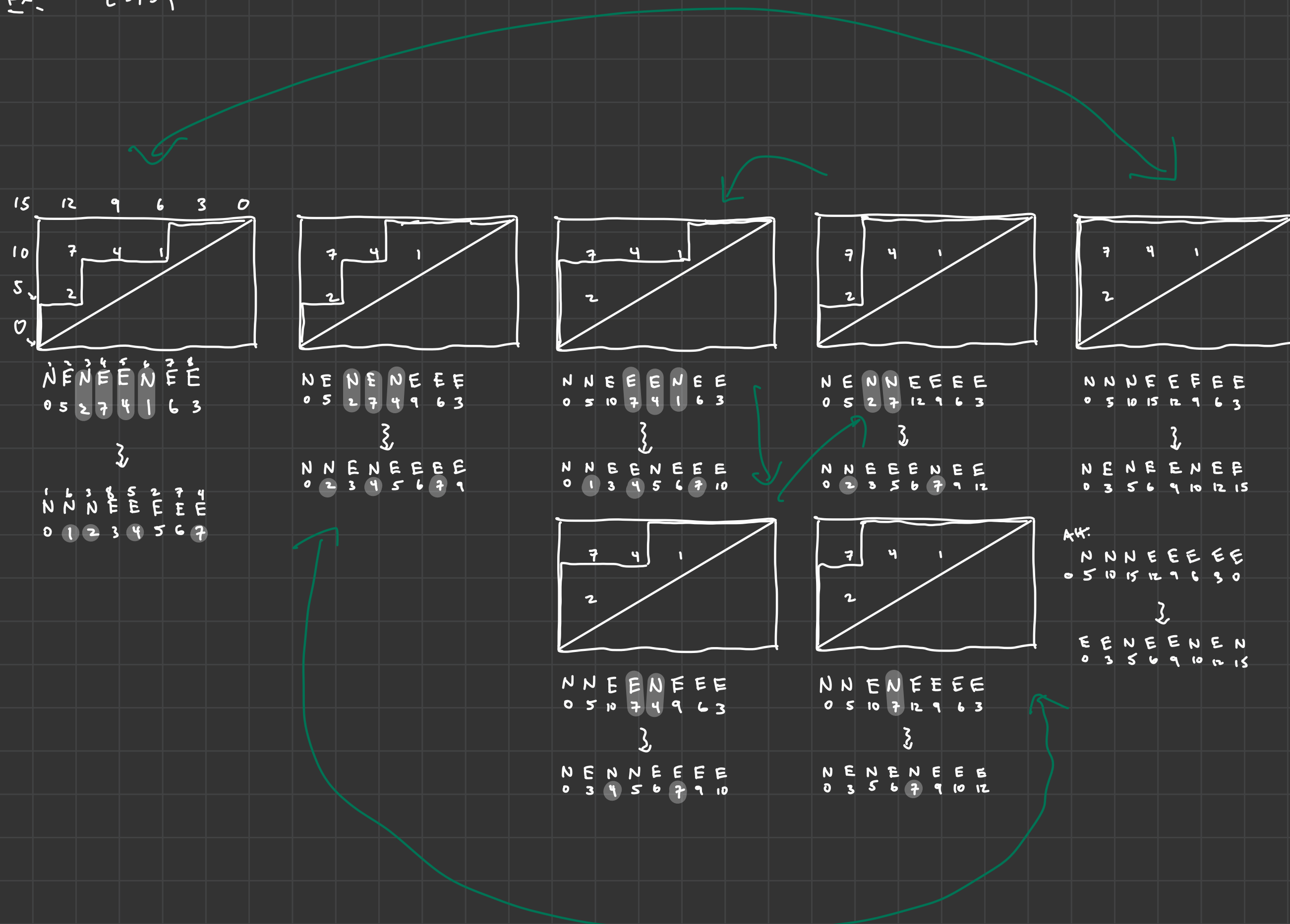
0 8 16 24 32 40 35 30 25 20 15 10 5



⇝
sweep



Ex. C(5,3)



15 12 9 6 3 0
10
5
0
7 4 1
2
N E N E E N E E
0 5 2 7 4 1 6 3
N N N E E E E E
0 1 2 3 4 5 6 7
N E N E N E E E
0 5 2 7 4 9 6 3
N N E N E E E E
0 2 3 4 5 6 7 9
N N E E E N E E
0 5 10 7 4 1 6 3
N N E E N E E E
0 1 3 4 5 6 7 10
N E N N E E E E
0 5 2 7 12 9 6 3
N N E E E N E E
0 2 3 5 6 7 9 12
N N N E E E E E
0 5 10 15 12 9 6 3
N E N E E N E E
0 3 5 6 9 10 12 15
N N E E N E E E
0 5 10 7 4 9 6 3
N E N N E E E E
0 3 4 5 6 7 9 10
N N E N E E E E
0 5 10 7 12 9 6 3
N E N E N E E E
0 3 5 6 7 9 10 12
Alt:
N N N E E E E E
0 5 10 15 12 9 6 3 0
E E N E E N E N
0 3 5 6 9 10 12 15

q,t-rational Catalan polynomials properties

Thm (W10 & when2) $C_{a,b}(q,t) = C_{a,b}(t,q)$

Cor. $C_{a,b}(q,1) = C_{b,a}(1,q)$

Q: Is there analogous theorem for $C_{a,b}(q,\frac{1}{q})$?

q-analogues:

$$C_{n,n+1}(q,1) = \sum_{D \in \mathcal{D}_{n,n+1}} q^{\text{area}(D)} \quad , \quad q^{?} C_{n,n+1}\left(q,\frac{1}{q}\right) = \frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n \end{bmatrix}_q$$

where $[n]_q = 1 + q + q^2 + \cdots + q^{n-1}$

$[n]_q! = [n]_q [n-1]_q \cdots [1]_q$