1 Logistics

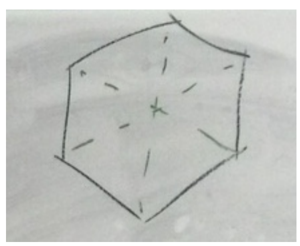
Next meeting: Friday February 15, 9:30 - 11am.

Expectations for next meeting:

- Problem 4: for the space of fold configurations for 3 creases (or 6 creases?) through a hexagon, what does a small neighborhood of the unfolded hexagon look like? Read "Branches of Triangulated Origami" paper (first 5 pages) to understand the answer given there.
- What do neighborhoods of other flat configurations look like? (How many flat configurations are there?) How do these neighborhoods fit together? An answer for the local neighborhoods is given in the paper "Hodge Theory and the Art of Paper Folding"; read pages 3–4 and the first page or so of Section 3 (p. 9) and Section 4 (p. 12).
- [**Writing**] Continue writing up relevant discussion from this week in draft of final report (e.g. What coordinates do we use for configuration space, fold angles vs. crease vectors?)
- [**Visualization**] Share code for visualizing space of fold configurations.

2 Moduli spaces

Problem 4. What is the space of folding configurations of a hexagon with creases along its three axes?



Discussion: (For now we consider just the massless case.) To describe a folding configuration we have different options for what "coordinates" to use:

(1) Fold angles $\theta_1, \ldots, \theta_6 \in \mathbb{R}/2\pi\mathbb{Z}$

($\mathbb{R}/2\pi\mathbb{Z}$ = a circle made by gluing together endpoints of a segment of length 2π)

(2) Crease vectors $u_1, \ldots, u_6 \in \mathbb{R}^3$

In case (1), our configuration space of all fold configurations is a subspace of $(\mathbb{R}/2\pi\mathbb{Z})^6 = (\mathbb{R}/2\pi\mathbb{Z}) \times \cdots \times (\mathbb{R}/2\pi\mathbb{Z})$. For creases in a unit hexagon it is hard(?) to explicitly describe which fold angles are allowed.

In case (2), our configuration space is a subspace of $(\mathbb{R}^3)^6$, modulo the action of rotations of $\mathbb{R}^3$ acting on all six vectors simultaneously. For the six creases on a unit hexagon, vectors $u_1, \ldots, u_6$ define a valid fold configuration if and only if these constraints are satisfied:

- $|u_i| = 1$ for $i = 1, \ldots, 6$
- $|u_i - u_{i+1}| = 1$ for $i = 1, \ldots, 6$ (where $u_7 := u_1$)

Once we observe this is a complete set of constraints, we can estimate the degrees of freedom in our configuration space by the heuristic

$$(\text{degrees of freedom}) = (\text{total dimension}) - (\#\text{ of constraints}).$$

3 Gaussian curvature

On a polyhedral surface we define the *Gaussian curvature* κ at a point x as

$$\kappa(x) = 2\pi - (\text{sum of angles around } x).$$

The curvature at a point can be positive or negative or zero. Folding does not change the curvature of a surface.

The curvature is defined as a local property ($\kappa(x)$ depends only on an arbitrarily small neighborhood of x), but there is an amazing theorem which relates the local curvature to the global topology of the surface!

Theorem (Gauss–Bonnet)**.** (a) Suppose S is a polyhedral surface which is homeomorphic to a 2-sphere. Then

$$\sum_{x \in S} \kappa(x) = 4\pi.$$

(In particular, $\kappa(x)$ is nonzero only at finitely many points of S, i.e. only when x is a vertex.)

(b) Suppose S is a closed polyhedral surface. Then

$$\sum_{x \in S} \kappa(x) = 2\pi\chi(S)$$

where χ is the *Euler characteristic* of the surface. A "g-holed torus" has $\chi(S) = 2 - 2g$.