1 Logistics

Make sure to send your Log(M) expectations form to Harry Bray!

I forgot to mention today: in addition to the midsemester presentation and final poster, you will be required to write up a report of your research in article format. We can discuss details at our next meeting.

Next meeting: Friday January 25, 10 - 11am.

2 Moduli spaces

A *moduli space* or *parameter space* is a (topological / geometric) space which parametrizes some family of objects. As sets, we at least want a bijection

$$\{\text{points in moduli space}\} \cong \{\text{objects in family}\}.$$

Often this "family of objects" means the possible configurations of a single underlying object which has some amount of flexibility. For this project we are interested in the possible configurations of a folded, rigid sheet, assuming that the folds must lie along a fixed set of creases.

Example 1. Consider a rigid sheet with a single crease (extending completely across the sheet). A folding configuration of this object is specified by a single angle θ. The moduli space of folding configurations is the set of allowed fold angles.

What counts as "allowed," and the resulting topology, depends on our interpretation:

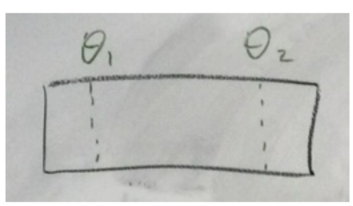
(a) If we assume the sheet cannot pass through itself and cannot fold completely flat (but arbitrarily close to flat), then the possible fold angles are $\{\theta : 0 < \theta < 2\pi\}$. Geometrically, this is an *open segment.*

If the sheet cannot pass through itself but can fold completely flat, then the allowed angles are $\{\theta : 0 \leq \theta \leq 2\pi\}$. Geometrically, this is a *closed segment.*

(b) If the sheet is allowed to pass through itself, we can have any angle including $\theta = 0$ and $\theta = 2\pi$ but these two angles describe the *same configuration.* (Why?) Thus the moduli space is the space $\{\theta : 0 \leq \theta \leq 2\pi\}/(0 \sim 2\pi)$ where we "glue together" angles 0 and 2π. Geometrically, this is a *circle.*

To have some consistent terminology, we can refer to folding models which are allowed to pass through themselves as "massless" and folding models which cannot pass through themselves as "massive". (Do you have other / better suggestions for what to call these?)

Example 2. Consider a rigid sheet with two parallel creases, which are far apart.



In this case the fold configuration is described by an ordered pair of angles (θ_1, θ_2).

(a) ["massive" case] If the sheet cannot pass through itself, the possible angles are

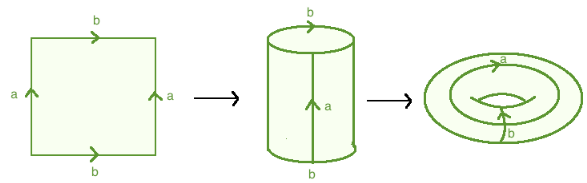
$$\{\theta_1, \theta_2 : 0 \leq \theta_1 \leq 2\pi,\, 0 \leq \theta_2 \leq 2\pi\}.$$

The space of fold configurations is a *square.*

(b) [“massless” case] If the sheet can pass through itself, the possible angles are

$$\{\theta_1, \theta_2 : 0 \leq \theta_1 \leq 2\pi,\ 0 \leq \theta_2 \leq 2\pi\}/(\,(0,\theta_2) \sim (2\pi,\theta_2),\ (\theta_1,0) \sim (\theta_1,2\pi)\,),$$

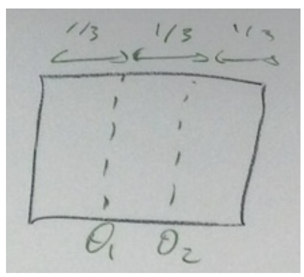
i.e. setting $\theta_1 = 0$ is the same as $\theta_1 = 2\pi$, regardless of θ_2, and vice versa. The space of fold configurations is a 2-*dimensional torus*, which we can construct by gluing together opposing edges of a square (without “twisting”).



(Image from `http://mathonline.wikidot.com/identifying-polygons-and-their-quotient-spaces`.)

For the following problems, consider both the “massive” case and the “massless” case.

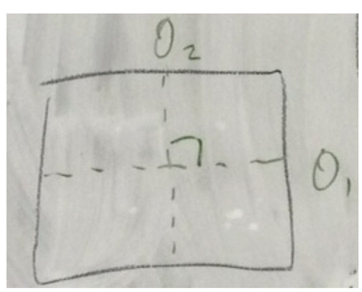
Problem 1. (a) What is the moduli space of folding configurations for two parallel creases, parallel to opposite sides, which split the sheet into 3 regions of equal area?



(b) What is the moduli space of folding configurations for two parallel creases, parallel to opposite sides, which are spaced arbitrarily?

(c) What distance is needed between the creases for this moduli space to look like Example 2 (the “far away” case)? What happens as the distance separating the two creases approaches 0?

Problem 2. (a) What is the moduli space of folding configurations for two perpendicular creases?



(b) What is the moduli space of folding configurations for two creases which cross in the center at an arbitrary angle? (Assume the creases intersect in the interior of the sheet)?

(c) Does your answer to (a) change if the creases are not aligned with the sides of the sheet? Does your answer to (a) or (b) change if the creases don’t intersect at the center?

Problem 3. [non-folding example of a moduli space] What is the moduli space of "space quadrilaterals" with fixed side lengths? E.g. if I cut four lengths of straw (with four different colors) and connect them in a loop with string, what are all possible configurations of this object (up to rotations and translations in 3-space)?

(Thanks to Alex Wright for the straw polygons!)

Expectations for next meeting:

- I think it should be possible to give a complete answer to Problems 1(a) and 2(a) (remember to include BOTH "massive" and "massless" cases), and Problem 3. Once you finish these, see if you can answer parts 1(b-c) and 2(b-c) as well.
- In the "Discrete folding" paper, try to understand Figure 9 on p. 10. How are the diagrams in the top row special? Where do the numbers next to each diagram come from? Is the model considered here "massless" or "massive"?

(Let me know if these expectations are not aligning with the hours of work we agreed to!)