Origami on Lattices

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I Introduction

Our goal for this project was classifying all origami configurations of a regular hexagon with six standard creases. More specifically we were interested in figuring out the degrees of freedom and the quantitative relationships between fold angles.

a A first example

We started our work by focusing on an easier example, to better understand the relationship between the moduli space of a configuration, and its degrees of freedom.

We considered a rigid sheet with two perpendicular creases that cross in the center of the sheet. The moduli space of such configurations has exactly 1 degree of freedom.

In the massive case, the only allowed combinations of angles are those where one of the angles is either 0, π, or 2π. The second angle can be anything. Any pair of angles that don't contain either 0, π, or 2π within the pair violates the constraint that the paper cannot pass through itself. One thing that should be noticed is that the four corner points, (0,0), (0, 2π),(2π, 0), (2π, 2π), represent 2 cases separately, indicating the different order of the overlapping. The moduli space is as shown in Figure1.

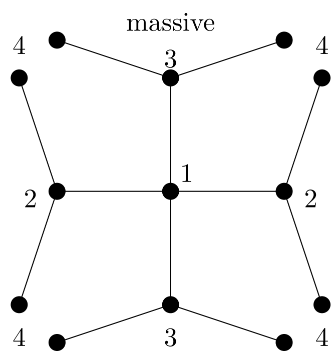


Figure 1: Massive moduli space

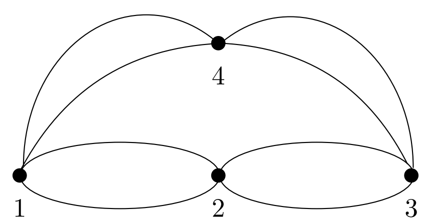


Figure 2: Massless moduli space

In the massless case, some of the allowed combinations of angles will represent the same configuration. The "duplicate" configurations in the massless case, were numbered in the massive moduli space figure. Thus, for the massless moduli space we connected the nodes corresponding to "duplicate configurations", as shown in Figure 2.

II Folding configuration of hexagon

Next we started investigating the space of folding configurations of a hexagon with creases along its three axes. In our prior example, the folding configurations were described by the fold angles. In the hexagon's case we decided to describe the folding configuration using two coordinate types:

(a) Fold angles $\theta_1, \ldots, \theta_6 \in \mathbb{R}/2\pi\mathbb{Z}$, with the configuration space being a subspace of $(\mathbb{R}/2\pi\mathbb{Z})^6$

(b) Crease vectors $u_1, \ldots, u_6 \in \mathbb{R}^3$ the configuration space is a subspace of $(\mathbb{R}^3)^6$

Due to the nature of the folds encountered in a hexagon, we realised that identifying the restrictions on the fold angles might not give us enough insight into the configuration space. Thus, we decided to firstly focus on describing it through crease vectors, and identified the two constraints on them, assuming the hexagon has unit length size:

- $|u_i| = 1$ for $i = 1, \ldots, 6$
- $|u_i - u_{i+1}| = 1$ for $i = 1, \ldots, 6$ (where $u_7 := u_1$)

By using crease vectors and the constraint on them, we deduced a formula describing the maximum degrees of freedom of our hexagon configuration. Since the configuration space of the hexagon is a subspace of $(\mathbb{R}^3)^6$, we know that the total dimension of the subspace is $6 * 3 = 18$. Since each of the six vectors has two constraints, the total number of constraints is 12. We consider configurations to be equivalent modulo rotations in $\mathbb{R}^3$. So the dimension of the symmetries is 3. From these observations, we obtain the equation:

$$\text{degrees of freedom} = (\text{total dim.}) - (\#\text{ constraints}) - (\text{dim. of symmetries}) = 6 \cdot 3 - 12 - 3 = 3$$

So the maximum number of degrees of freedom is 3.

To study the configurations space, we will look at it both locally (defined as near the unfolded configuration) and globally.

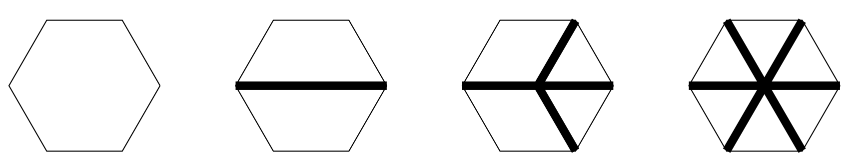
a Local Structure

We used the paper *Branches of Triangulated Origami Near the Unfolded State*[1] to get the geometry of the nearly unfolded 6-fold single-vertex origami. The paper gave us the relationship:

$$\sum_{n,m} Q_{nm} h_n h_m = 0 \tag{1}$$

where Q_{nm} is the symmetric matrix corresponding to ω a stress vector with one component per fold, and h_n, h_m represent the vertical displacements out of the plane (where n and m denote vertices). Based on the paper's results, since we have 6 folds, we know that the 7x7 matrix Q_{nm} has 4 nonzero eigenvalues. The solutions to Eq (4), define a 3 dimensional space, known as the null cone of this quadratic form. Thus, the origami configuration space near a singular point is described by the null cone of a quadratic form.

Bowick et al.(1996)[3] showed that the singular points of the hexagon configuration space are the following flat configurations, with their corresponding numbers of possible permutations:



1 3 6 1

Since the singular points of the hexagon configuration space lie in a great circle S^1 of S^2 , we know that they are degenerate n-gon linkages[2]. After orienting S^1, the i-th edge of a singular point is forward-track if its orientation agrees with that of S^1, and backward-track otherwise. We let f be the number of forward-track edges, and b be the number of backward-track edges. We define w as $\sum_{i=1}^{n} \epsilon_i r_i = 2\pi w$. To get a geometrical description of the null cones associated with the configurations at the near-unfolded state of each of these flat configurations, we introduced the idea of a quadratic form's signature defined as $(f - 2w - 1, b + 2w - 1)$[2].

We get that for the unfolded configuration we have f=6, b=0, and w=1. The corresponding null cone has signature (3, 1). For the other 10 singular points, f=3, b=3 and w = 0, so the corresponding null cone has signature (2, 2).

Note that the quadratic form of the null cone of signature (3,1) is $\{(a, b, c, d) \in \mathbb{R}^4 : a^2 + b^2 + c^2 - d^2 = 0\}$, and that of the null cone of signature (2,2) is $\{(a, b, c, d) \in \mathbb{R}^4 : a^2 + b^2 - c^2 - d^2 = 0\}$. If we consider restricting these null cones such that $(a, b, c, d) \in \mathbb{R}^4$ has unit length, we get a *unit null cone*. This leads us to an important theorem.

Theorem 1 *The unit null cone of a form of signature (p,q) is a product of spheres* $S^{p-1} \times S^{q-1}$

Thus, the unfolded configuration is geometrically described by the unit null cone $S^2 \times S^0$, as shown in Figure 3, and the other 10 flat configurations are described by the unit null cone $S^1 \times S^1$, as shown in Figure 4.

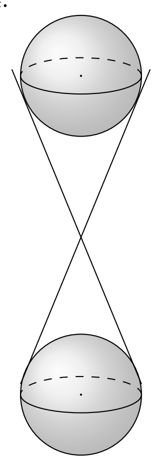


Figure 3: Unit Null cone for signature (3,1)

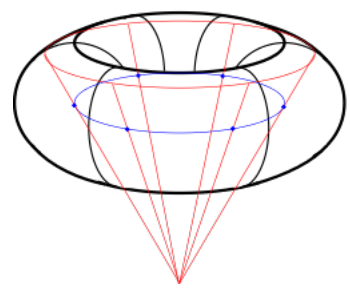


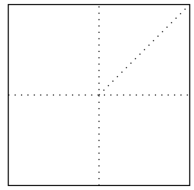
Figure 4: Unit Null cone for signature (2,2)

III Mountain / valley regions near unfolded state (signature $(3, 1)$ null cone)

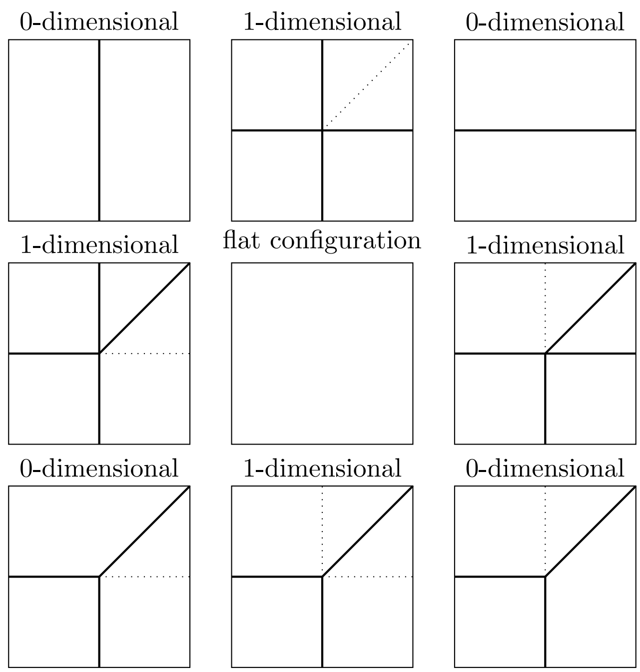
To better understand the topological space of the flat configuration with corresponding signature (3,1), we decided to cut it up into smaller, more manageable pieces. We defined a crease as a mountain fold if it is higher than a secant line between its two adjacent flat regions, and valley fold if lower. Visually, we labeled mountain, valley, and flat folds respectively as a solid line, a dotted line, and no line. With these definitions we can describe the topological space near the singular points.

a Mountain/valley labellings for a simpler case

To gain a better understanding of the properties we would encounter when trying to describe the configuration space near the unfolded state through mountain/valley regions, we looked at an easier example than that of a hexagon. We considered the case of a rigid sheet with two creases perpendicular to one another, and one crease at 45°, as shown below:

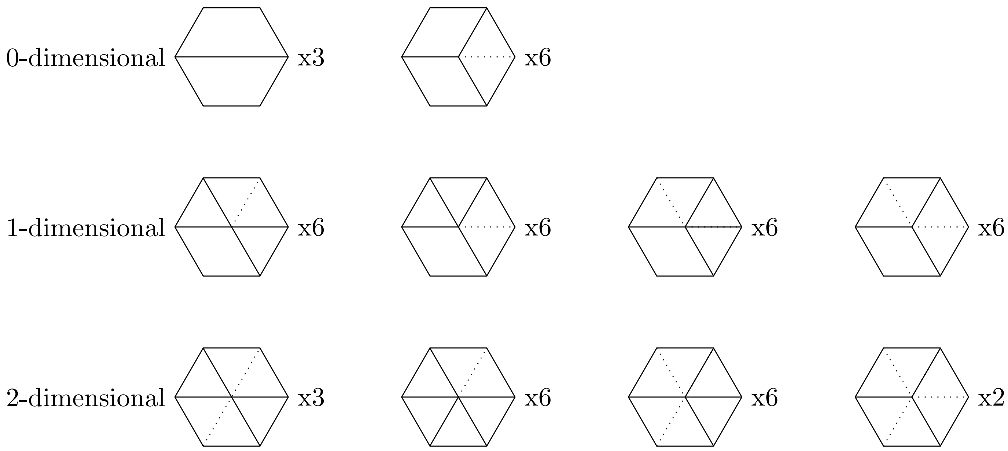


By establishing all the mountain/valley regions we realized that their degree of freedom varied. To define this differentiating feature, we defined the idea of dimensions as (# degrees of freedom) − 1. We found that there are 4 one dimensional regions and 4 0-dimensional regions.

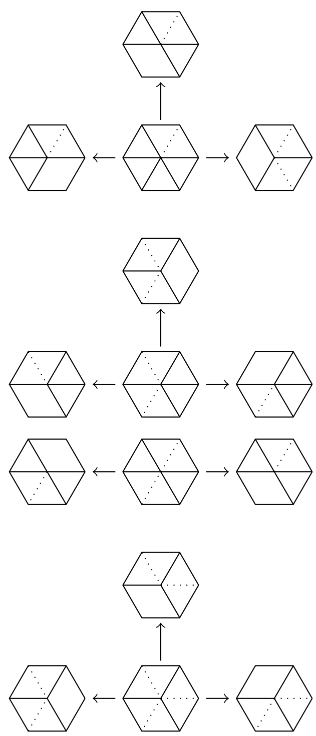


b Mountain/valley labellings for hexagon

Having defined the concept of dimensions for the mountain/valley regions in the null cone space, we identified the regions associated to the valid folding configurations of the hexagon. Dividing the space into mountain/valley labellings, we got nine 0-dimensional, twenty-four 1-dimensional, and seventeen 2-dimensional configurations.



When fitting these regions together to form a topological subspace of the unit null cone, we look at the neighbouring configurations for each of the 2-dimensional and 1-dimensional regions.We know that every 2-dimensional configuration has three 1-dimensional configurations as neighbours (except for one 2-dimensional configurations that only have two 1-dimensional neighbours), and each one of those has two 0-dimensional configurations as neighbours. Below, we have four of the 2-dimensional configurations, with arrows pointing to their 1-dimensional neighbours:



If we consider the 0-dimensional configurations to be edges, the 1-dimensional configurations to be vertices and the 2-dimensional configurations to be faces, then we get 9 edges, 24 vertices, and 17 faces. This satisfies Euler's Formula:

$$V - E + F = 2$$

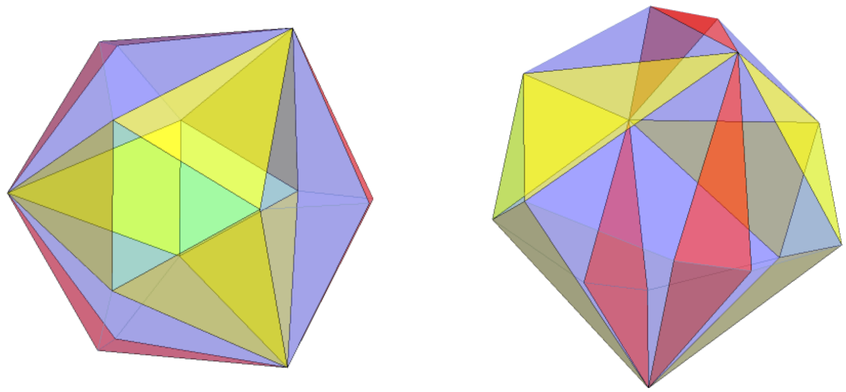


Figure 5 :Polyhedron visualization of hexagon configurations

Thus, by knowing the neighbours we can assign the mountain/valley configurations to our polyhedron's vertices, edges, and faces. Flattening our polyhedron gives us a better view of the positioning of the configurations:

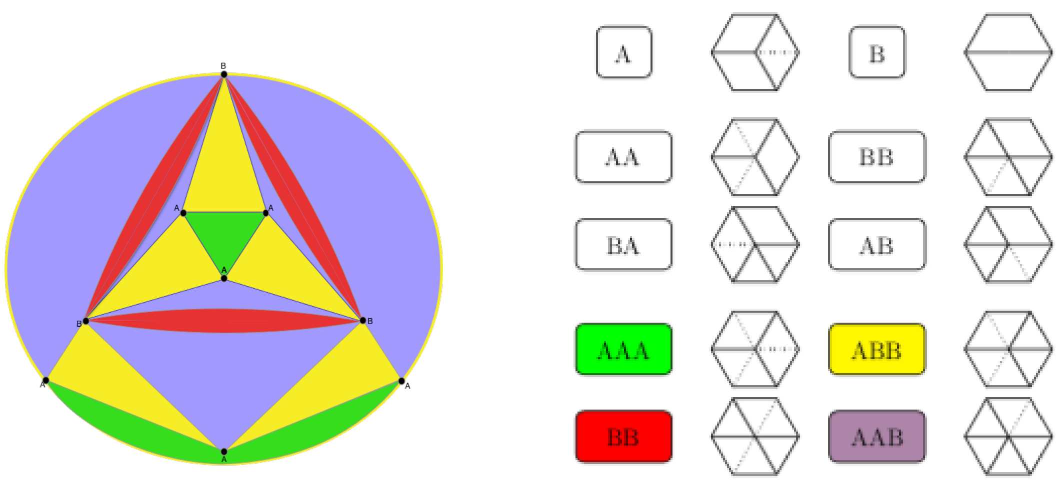
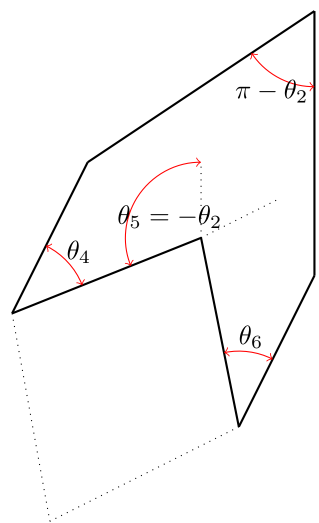


Figure 6: Associated mountain-valley configurations

IV Angle formulas in specific mountain / valley regions

Based on previous near flat configurations, we calculated the quantitative restrictions in terms of fold angles. Each crease is associated with one fold angle, where 0 represents flat. A fold angle measures how far the sides of the creases are from the flat position. Thus, it is equal to $\pi-$ {dihedral angle between two neighboring equilateral triangles}. The range of a fold angle is $[\pi, \pi]$.

Suppose $\theta_1 = \theta_3 = 0$ and θ_2 is arbitrary. Then one possibility is that $\theta_4 = \theta_6 = 0$ and $\theta_5 = \theta_2$. A second possibility is that crease 5 is flipped inward, so that $\theta_5 = \theta_2$, and θ_4 and θ_6 are at some positive angle. In this case, we get the view from above of the configuration shown below:



Thus, we can trace an imaginary rhombus with angles $\pi - \theta_4$ and $\pi + \theta_5 = \pi - \theta_2$. Let $\pi - \theta_4 = \alpha$ and $\pi + \theta_5 = \beta$. Then, we know from spherical trigonometry that:

$$\cos\frac{\alpha}{2} + \cos\frac{\alpha}{2}\cos\beta = \sin\frac{\alpha}{2}\sin\beta cos L$$

$$\cos\frac{\alpha}{2}(1 + \cos\beta) = \sin\frac{\alpha}{2}\sin\beta\cos L$$

$$\cos\frac{\alpha}{2}(1 + \cos^2\frac{\beta}{2} - 1) = \sin\frac{\alpha}{2}2\sin\frac{\beta}{2}\cos\frac{\beta}{2}\cos L$$

$$\cos\frac{\alpha}{2}\cos\frac{\beta}{2} = \sin\frac{\alpha}{2}2\sin\frac{\beta}{2}\cos L$$

$$\tan\frac{\alpha}{2}\tan\frac{\beta}{2} = \frac{1}{\cos L}$$

$\theta_4 = \theta_6$ can be written as a function of θ_2 as:

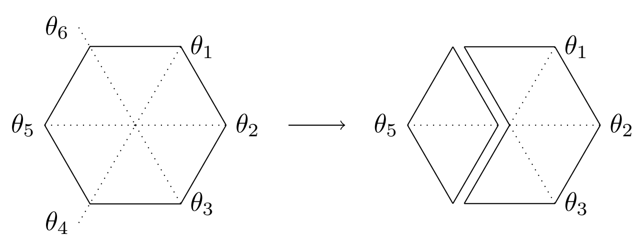
$$\tan\frac{\pi - \theta_2}{2}\tan\frac{\pi - \theta_4}{2} = \frac{1}{\cos L}$$

V Distance formula in terms of $\theta_1, \theta_2, \theta_3$

Each of our fold angles is in $[-\pi, \pi]$ so the configuration space of a single hexagon is a bounded space in $[-\pi, \pi]^6$ hypercube. We can project this space to $[-\pi, \pi]^3$ by only considering three of the six angles, based on the fact that our hexagon has a maximum of 3 degrees of freedom.

Divide and Bound Our goal is to generate a boundary restriction relationship by taking three consecutive fold angles as variables.

Divide We divide the hexagon into two parts: a rhombus consisting of two neighbouring equilateral triangles, and a 6-crease concave polygon.



Bound Given the fold angle θ_5, we can calculate the restrictions on the distance between vertices 4 and 6 (where each vertex corresponds to the appropriately numbered fold angle). To have a valid configuration, the distance must be within the interval $[0, \sqrt{3}]$. Correspondingly, the distance in terms of θ_1, θ_2,θ_3 should take the same range. This also indicates we should have the cosine value of the angle between $\mathbf{v_4}$ and $\mathbf{v_6}$ in $[-\frac{1}{2}, 1]$.

$$Distance_{\mathbf{v_4},\mathbf{v_6}}(\theta_5) = \sqrt{\frac{3}{2} - \frac{3\cos\theta_5}{2}} \in [0, \sqrt{3}] \Rightarrow Distance_{\mathbf{v_4},\mathbf{v_6}}(\theta_1, \theta_2, \theta_3) \in [0, \sqrt{3}]$$

$$\Leftrightarrow \cos(\angle(\mathbf{v_4}, \mathbf{v_6})) = \mathbf{v_4} \cdot \mathbf{v_6} \geq -\frac{1}{2}$$

We represent the relationship of $\mathbf{v_4}$ and $\mathbf{v_6}$ through rotation transformation calculations. Given $\mathbf{v_6}$, we consider $\mathbf{v_1}$ as the transformation of $\mathbf{v_6}$ rotating by $\pi/3$ around z-axis and then rotating by $\pi - \theta_1$ around former $\mathbf{v_6}$-axis. By the same manner, we can get $\mathbf{v_4}$ by rotating through $\mathbf{v_1}$, $\mathbf{v_2}$, $\mathbf{v_3}$ in clockwise direction.

To get six triangles to form a hexagon in the xy-plane, we can start with one equilateral triangle with a vertex at the origin and then rotate it by angle $\pi/3$ around the z-axis five times, to get the other five triangles. This generates an unfolded hexagon. Let $F(0)$ denote the *(hexagon) fold matrix*

$$F(0) = \begin{pmatrix} \cos\frac{\pi}{3} & -\sin\frac{\pi}{3} & 0 \\ \sin\frac{\pi}{3} & \cos\frac{\pi}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

To get a *folded* hexagon, we can start with the same triangle in the xy-plane whose upper edge is along the x-axis, and apply another rotation around the x-axis. The angle of the x-axis rotation is the *fold angle*, and the corresponding *fold matrix* is

$$F(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2}\cos\theta & \frac{1}{2}\cos\theta & -\sin\theta \\ \frac{\sqrt{3}}{2}\sin\theta & \frac{1}{2}\sin\theta & \cos\theta \end{pmatrix}$$

Given $\mathbf{v_6} = (\frac{\sqrt{3}}{2}, \frac{1}{2}, 0)$, without loss of generality, we take $\mathbf{v_1} = (1, 0, 0)$. Then we obtain $\mathbf{v_2}$ by the transformation matrix $F(\theta_1)$, $\mathbf{v_2} = F(\theta_1)\mathbf{v_1}$. For $\mathbf{v_3}$, since we need to change the base of rotation, the transformation matrix from $\mathbf{v_1}$ to $\mathbf{v_3}$ is

$$M_{\mathbf{v_1},\mathbf{v_3}} = F(\theta_1)F(\theta_2)F^{-1}(\theta_1) \Rightarrow \mathbf{v_3} = M_{\mathbf{v_1},\mathbf{v_3}}\mathbf{v_1} = M_{\mathbf{v_1},\mathbf{v_2}}F(\theta_1)\mathbf{v_1} = F(\theta_1)F(\theta_2)\mathbf{v_1}.$$

The matrix product $F(\theta_1)F(\theta_2)$ sends the original triangle to the "next-next" triangle in a hexagon with fold angles θ_1 and θ_2. Similarly, we get $\mathbf{v_4} = F(\theta_1)F(\theta_2)F(\theta_3)\mathbf{v_1}$.

$$\mathbf{v_4} \cdot \mathbf{v_6} = F(\theta_1)F(\theta_2)F(\theta_3)\mathbf{v_1} \cdot \mathbf{v_6} \geq -\frac{1}{2} \Leftrightarrow$$

$$\begin{aligned} -\frac{1}{2} \leq \frac{1}{16}(&1 - 3\cos\theta_1 - 3\cos\theta_2 - 3\cos\theta_3 - 3\cos\theta_1\cos\theta_2 - 3\cos\theta_2\cos\theta_3 \\ &+9\cos\theta_1\cos\theta_3 - 3\cos\theta_1\cos\theta_2\cos\theta_3 + 6\sin\theta_1\sin\theta_2 + 6\sin\theta_2\sin\theta_3 \\ &+6\cos\theta_1\sin\theta_2\sin\theta_3 + 6\sin\theta_1\sin\theta_2\cos\theta_3 + 12\sin\theta_1\cos\theta_2\sin\theta_3) \end{aligned}$$

Visualization and Interpretation The configuration space defined by the above equation is represented in Figure 7. On its surface, θ_5 is 0. Figure 8 is the representation of the configuration space when θ_1, θ_2, θ_3 are near 0. The corresponding contours consist of four 1-dimensional configurations where $\theta_5 = 0$ as shown in Figure 9.

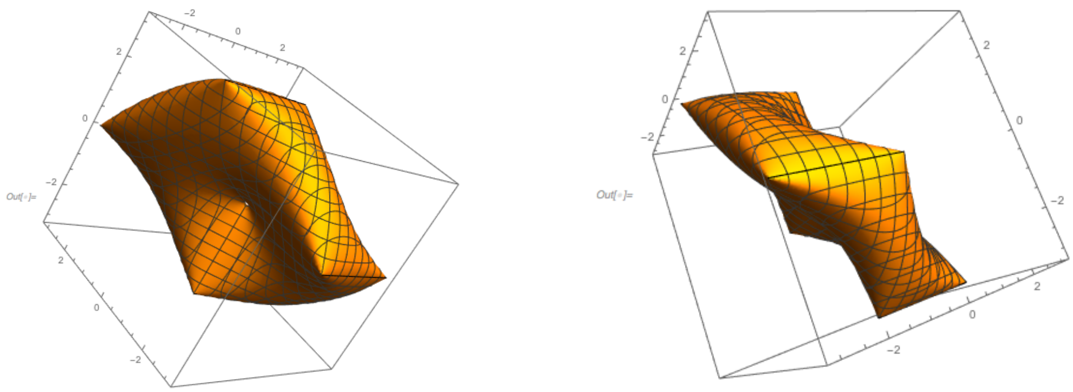


Figure 7: Configuration space boundary of hexagon

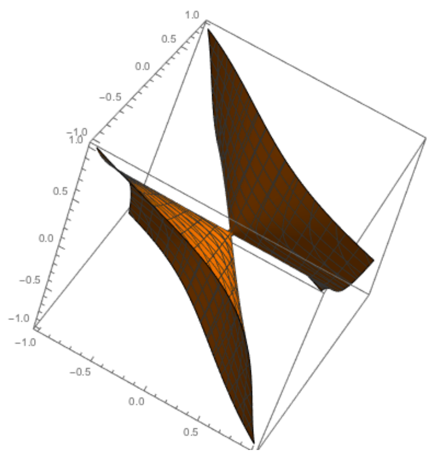


Figure 8: Boundary near unfolded configuration

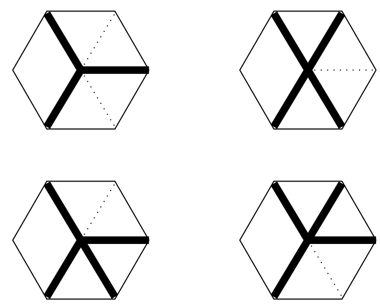


Figure 9: 4 1-dimensional components of near unfolded configurations

VI Energy propagation

Using the energy function defined by $E(\Phi) = (\sum |\theta_i|^2)^{1/2}$ and our quantitative constraints on the hexagon, suppose we fix fold angle θ_5 at some nonzero value $\theta_5 > 0$. The rest of the hexagon cannot lie completely flat. We would like to quantify this by finding some constant C such that for any fold configuration $E(\theta_1, \dots, \theta_6) \geq C\theta_5^2$ With the constraint, we use this approximation for lower bound of energy:

$$E(\theta_1, \theta_2, \theta_3) = \theta_1^2 + \theta_2^2 + \theta_3^2$$

In the context of near flat configurations, θ_1,θ_2,θ_3 are close to 0. We use cosine and sine Taylor expansions to simplify the constraint function to a sum of a quadratic function. We obtain

$$\mathbf{v_4} \cdot \mathbf{v_6} \approx \frac{1}{16}(1 - 3(1 - \frac{\theta_1^2}{2}) - 3(1 - \frac{\theta_2^2}{2}) - 3(1 - \frac{\theta_3^2}{2}) - 3(1 - \frac{\theta_1^2}{2})(1 - \frac{\theta_2^2}{2})$$

$$-3(1 - \frac{\theta_2^2}{2})(1 - \frac{\theta_3^2}{2}) + 9(1 - \frac{\theta_1^2}{2})(1 - \frac{\theta_3^2}{2}) - 3(1 - \frac{\theta_1^2}{2})(1 - \frac{\theta_2^2}{2})(1 - \frac{\theta_3^2}{2})$$

$$+6\theta_1\theta_2 + 6\theta_2\theta_3 + 6\theta_2\theta_3(1 - \frac{\theta_1^2}{2}) + 6\theta_1\theta_2(1 - \frac{\theta_3^2}{2}) + 12\theta_1\theta_3(1 - \frac{\theta_2^2}{2}))$$

$$\mathbf{v_4} \cdot \mathbf{v_6} \approx f_q(\theta_1, \theta_2, \theta_3) = -\frac{1}{2} + \frac{3}{8}(2\theta_1\theta_3 + 2\theta_2\theta_3 + 2\theta_1\theta_2 + \theta_2^2)$$

By the method of Lagrange multipliers, we have

$$\begin{cases} q(\theta_1, \theta_2, \theta_3) = 2\theta_1\theta_3 + 2\theta_2\theta_3 + 2\theta_1\theta_2 + \theta_2^2 \\ \varphi_(\theta_1, \theta_2, \theta_3) = \theta_1^2 + \theta_2^2 + \theta_3^2 \\ q(\theta_1, \theta_2, \theta_3) = \theta_5^2 = \boldsymbol{\epsilon} \end{cases}$$

$$\bigtriangledown q = (2\theta_1, 2\theta_2, 2\theta_3)$$

$$\bigtriangledown f = (2\theta_2 + 2\theta_3, 2\theta_1 + 2\theta_2 + 2\theta_3, 2\theta_1 + 2\theta_2$$

$$\bigtriangledown f = \lambda \bigtriangledown f$$

$$\Rightarrow \lambda = 1, 1 - \sqrt{2}, 1 + \sqrt{2}$$

With $q(\theta_1, \theta_2, \theta_3) \geq 0$, we determine $\lambda = 1 + \sqrt{2}$. Substitute $[\theta_1, \theta_2, \theta_3]^T = \boldsymbol{\alpha}[1, \lambda - 1, -1]^T$ back, we got

$$\boldsymbol{\epsilon} = 4(\sqrt{2} + 1)\boldsymbol{\alpha}^2$$

$$\varphi = \boldsymbol{\alpha}^2[1, \sqrt{2}, -1]^2 = 4\boldsymbol{\alpha} = \frac{\boldsymbol{\epsilon}}{1 + \sqrt{2}}$$

$$\Rightarrow C = \frac{1}{1 + \sqrt{2}}$$

References

[1] Chen, B. G. & Santangelo, C. D. *Branches of Triangulated Origami Near the Unfolded State* Physical Review X. 8, 2018.

[2] Kapovich, M. & Millson, J. J. *Hodge Theory and the Art of Paper Folding.* Publications of the Research Institute for Mathematical Sciences. 33, 1997.

[3] Bowick, M. J., Francesco P., Golinelli O. & Guitter, E. preprint 1996