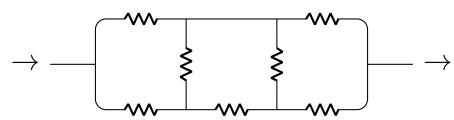
RESEARCH STATEMENT

DAVID HARRY RICHMAN

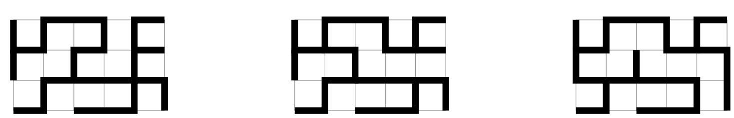
My research is in *tropical geometry*, *random graph theory*, *phylogenetics*, and *number theory*. In these areas, a central role is played by the notion of *effective resistance* of an electrical network. Early work on effective resistance was pioneered by Kirchhoff [Kir47] at the time when electricity was a new technology. As a motivating example, consider the following resistor network, where each resistor has unit resistance.



For electricity passing through this resistor network, how much effective resistance will be encountered? What fraction of the total current will pass through each individual wire in the network? Despite these old beginnings, effective resistance has found applications in cutting-edge research in arithmetic geometry [Zha93; CR93; KRZB16], probability theory [LP14; KW15] and theoretical computer science [SS11; Asa+17].

From the beginning, the practical desire to answer these questions resulted in a surprising and beautiful connection of algebra and combinatorics, known as Kirchhoff's *matrix tree theorem*. Kirchhoff found that effective resistances and current flows could be expressed through counting *spanning trees* of the underlying network. These spanning tree counts, in turn, could be found by taking determinants of certain matrices.

Random graph theory. Kirchhoff's work concerning effective resistance can be expressed in terms of the *uniformly random spanning tree* (UST), a random process that selects any spanning tree with equal probability. Three spanning trees in a grid graph are shown below.



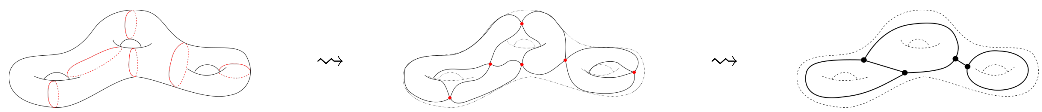
Effective resistance is fundamentally connected to another natural random process, the *simple random walk* on a graph [NW59; Tet91]. Aldous [Ald90] and Wilson [Wil96] showed how to generate a UST directly from a random walk. Wilson's method for generating a UST achieved improved algorithmic efficiency, by utilizing *loop erased random walks* (LERW) [Law91]. The study of electricity was also the initial inspiration for the notion of *capacity*. When electrically charged particles are introduced to a conductive material, say a metal plate, then where do they go? Kakutani [Kak49] showed that random walks solve the graph-theoretic analogue of this problem.

These statistics on spanning trees and spanning forests are also essential to calculations in string theory, where Feynman graphs describe particle interactions [Ami+16].

Date: February 28, 2024.

Email: `hrichman@alum.mit.edu`.

Tropical geometry. *Tropical geometry* forms a bridge between continuous objects of algebraic geometry and discrete objects of combinatorics. Algebraic geometry is the study of solutions to polynomial equations such as $x^4 + y^4 = 10$. Over the complex numbers, the set of solutions is known as a Riemann surface. Tropical geometry allows us to turn a Riemann surface into a graph, as shown below.



This can be achieved via *degenerating* a smooth algebraic curve to a curve with nodal singularities, along a one-parameter family, then taking the dual graph of the nodal curve. Tropical geometry has been used to solve difficult problems in number theory and algebraic geometry. For instance, it is believed that there are *uniform bounds on rational points* on curves of genus $g \geq 2$, strengthening Faltings' theorem, but no such bound is currently known. Recent work of Katz, Rabinoff, and Zureick-Brown [KRZB16] made progress toward such a bound, for curves which satisfy an additional assumption on the Mordell–Weil rank. Tropical geometry was a fundamental ingredient in their proof.

As an analogue to the conjecture that there exist uniform bounds on the number of rational points on curves, there was a long-standing conjecture that there are uniform bounds on the number of torsion points in the Jacobian of an algebraic curve. This conjecture was recently answered, in the affirmative, by [LSW21; Küh21; DGH21]. For a survey of this recent work, see [Gao21].

This degeneration process turns meromorphic functions on the Riemann surface to piecewise linear functions on the dual graph. These tools were developed by Baker–Norine [BN07] and others [MZ08; GK08]. My research studies tropical analogues of theorems from algebraic geometry concerning special discrete subsets of algebraic curves, known as *Weierstrass points*.

When studying Weierstrass points on tropical curves, I discovered that the limiting distribution of these points tends to a pattern known as the *canonical measure* of the graph. This canonical measure was first studied in connection to arithmetic geometry (rational points on algebraic curves) by Zhang [Zha93]. The canonical measure can be defined using effective resistance [CR93].

Phylogenetics. Phylogenetics is the study of evolutionary histories of organisms. Given observed traits of modern-day organisms, how did they diverge over time from a common ancestor?

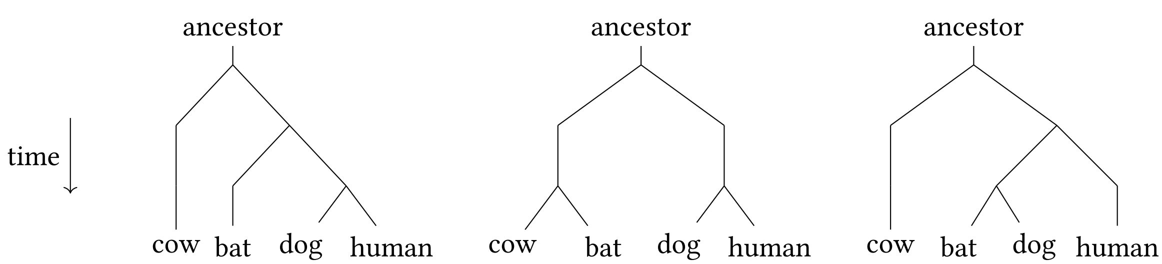


Figure 1. Possible phylogenetic trees on four species.

In its current form, the phylogenetic "observed traits" are DNA or protein sequences. (Historically, this was called *molecular phylogenetics* in contrast to the use of macroscopic observed traits.) "Solving" phylogenetic inference is computationally hard. It was shown to be NP-complete by Foulds and Graham [FG82] in the *maximum parsimony* framework. Phylogenetics research often focuses on heuristic for "guessing" better evolutionary trees.

In order to solve a phylogenetics problem with a computer, it is necessary to translate a tree into a computer-readable format. There are many ways to do this, where it is not obvious what advantages a given format has over another. In one project, I evaluated different tree formats using a *contrastive learning* framework. The goal is to find which format causes groups of plausible trees to be "clumped together." The "groups of plausible trees" comes from datasets collected by other researchers, using *Bayesian phylogenetics* which is typically computationally slow.

As part of this project, I introduced a new format to encode rooted, binary phylogenetic trees, called the *ordered leaf attachment (OLA) code*. This format has the potential to improve the efficiency of many phylogenetic computations. I investigated how the OLA-code-induced distance is connected to others commonly-used in phylogenetics, such as the *subtree-prune-regraft* (SPR) distance.

Number theory. A central problem in number theory is to understand the distribution of prime numbers. The *Riemann hypothesis* can be viewed as quantifying the "randomness" of the primes in some sense. This is done via bounding the growth of the partial sums of the Möbius function defined on the positive integers under divisibility. A common approach to the Riemann hypothesis is to try "deforming" the underlying structure, and testing to what extent other properties are preserved or broken. I take this approach of "deforming" the multiplicative structure of positive integers by allowing division with rounding, and study the resulting Möbius function.

1. Graph theory

1.1. **Random two-forests.**
In joint work with F. Shokrieh and C. Wu [RSW23], we prove results in graph theory via use of effective resistance and potential theory on graphs. In particular, we show bounds on the number of two-component spanning forests that generalize foundational results on lattices motivated by loop erased random walks [Law91]. The effective resistance can be expressed as a ratio of counts of spanning trees and certain *two-component spanning forests*, or "two-forests".

Theorem 1 ([RSW23])**.** *If F is a uniformly-random two-forest on $G = (V, E)$, then*

$$\mathbb{E}(|\partial F|) \leq 2(\text{avg.deg}) \left(1 + \frac{1}{|V| - 1}\right).$$

Here "avg. deg" is the average degree of the vertices, $2|E|/|V|$.

Theorem 2 ([RSW23])**.** *For any finite graph $G = (V, E)$,*

$$\frac{\kappa_2(G)}{\kappa_1(G)} \geq \frac{(|V| - 1)^2}{4|E|} \tag{1}$$

where κ_1 denotes the number of spanning trees and κ_2 the number of two-forests.

The looping constant of the loop-erased random walk (LERW) on a graph, can be linked to the behavior of a uniformly random two-forest. This motivates the study of random two-forests on planar lattices [KKW15; KW16; KW15; LP14]. In the case when G is taken to be a "large enough" subset of the lattice $\mathbb{Z}^d$, with nearest-neighbor edges, the bound in Theorem 1 is sharp.

Theorem 2 is related to *Mason's conjecture*, on the log concavity of matroid independence numbers. Mason's conjecture was solved by [Ana+18; BH20] in independent work, after several decades of research activity. The bound (1), in matroid theoretic language, involves independence numbers I_{r-1}/I_r and I_0/I_1 for graphic matroids; this bound is stricter than the one implied by Mason's conjecture. It would be interesting to study whether stronger bounds can be given on I_{k-1}/I_k for graphic matroids, for other k.

1.2. **Distance minors of trees.** In other joint work [RSW], we study a special case of effective resistance: on a tree, the resistance is equal to the usual shortest-path distance. Graham and Pollak [GP71] found that the determinant of the distance matrix reduces to a simple expression depending *only* on the number of vertices.

We extend their result by finding an expression for the determinant of an arbitrary principal minor of the distance matrix. Unlike in [GP71], this expression depends on combinatorics of the underlying graph and vertex subset, involving counts of rooted spanning forests and boundary edges.

Theorem 3 ([RSW])**.** *Suppose $G = (V, E)$ is a tree with distance matrix D, and let $S \subset V$ be a nonempty subset of vertices. Then the principal minor $D[S]$ has determinant*

$$\det D[S] = (-1)^{|S|-1} 2^{|S|-2} \left(|E| \cdot \kappa_1(G;S) - \sum_{\mathcal{F}_2(G;S)} (\deg^o(F,*) - 2)^2 \right).$$

This result is proved using methods from potential theory—we associate to the data (G, S) a certain function which is "extremal" for the energy functional, and then manipulate algebraically. Potential theory was initially developed as a tool for understanding the behavior of electric change in a material that is electrically conductive [Gre54]. Theorem 3 also has a generalization to graph with edges lengths. Bapat, Lal, and Pati [BLP06] found that the Graham–Pollak determinant formula generalizes to q-distance matrices (in two ways). We hope to investigate whether Theorem 3 also generalizes to q-analogues of distance.

We also hope that the expression in Theorem 3 may be used to make progress toward the *lower bound conjecture* for the tau constant on metric graphs [CR93].

2. Tropical geometry

A projective embedding of an algebraic curve is naturally associated with a family of divisors on the curve, by intersecting the embedded curve with hyperplanes of the ambient space. The *Weierstrass points* of a divisor are the points in the corresponding embedding in $\mathbb{P}^r$ where the curve intersects some hyperplane with "higher-than-expected" multiplicity. On a genus one curve, this condition for a degree n divisor gives a set of n-torsion points; thus Weierstrass points are a higher genus analogue of torsion points [Mum75].

2.1. **Tropical Weierstrass points.** In [Ric24b], I study a natural analogue of Weierstrass points for a tropical curve. In particular, the number of Weierstrass points for a generic divisor is determined as a function of the degree and genus; a limiting distribution is proved as the degree grows to infinity; weights are determined by a combinatorial formula which matches the number of algebraic Weierstrass points under inverse-tropicalization.

Theorem 4 ([Ric24b])**.** *On a metric graph of genus g, a generic divisor of degree $n \geq g$ has $g(n-g+1)$ Weierstrass points.*

Theorem 5 ([Ric24b])**.** *Let Γ be a metric graph of genus $g \geq 2$, and let δ_n be the unit discrete measure supported on the Weierstrass locus of a generic divisor of degree n. Then the sequence of normalized measures $\frac{1}{gn}\delta_n$ converges weakly to Zhang's canonical measure on Γ.*

The distribution result mirrors parallel results of Neeman [Nee84] and Amini [Ami14] for algebraic curves over the complex numbers $\mathbb{C}$ and over a field with non-Archimedean valuation, respectively. The Berkovich analytification of a curve [Ber90] contains a tropical curve as its *skeleton*, and divisor theory behaves well with respect to retraction to the skeleton [Bak08]. Amini's result suggested that the distribution of Weierstrass points could be a purely tropical phenomenon.

2.2. **Weierstrass weights.** Although [Ric24b] gives a fairly complete description of the tropical Weierstrass locus for a generic divisor, many divisors of particular interest are not generic. Most prominently, it does not address the tropical Weierstrass locus of the *canonical divisor* K. In joint work with O. Amini and L. Gierczak [AGR23], we develop techniques for addressing the tropical Weierstrass locus on an arbitrary divisor, including K. We consider how Weierstrass points on an algebraic curve, over a non-Archimedean field, tropicalize to its "skeleton" tropical curve.

Theorem 6 ([AGR23]). *Suppose $A \subset \Gamma$ is a closed, connected subset which is $W(K)$-measurable. Then, the total weight of Weierstrass points of $\mathcal{W}(\mathcal{K})$ tropicalizing to points in A is precisely*

$$\deg\left(\mathcal{W}(\mathcal{K})|_{\tau^{-1}(A)}\right) = g\Bigg((g+1)\,(g(A)-1) - \sum_{\nu\in\partial^{\rm out}A} (s_0^\nu(K)-1)\Bigg).$$

In particular, if K is Weierstrass-finite, then we have the equality $\tau_*\left(\mathcal{W}(\mathcal{K})\right) = g\,W(K)$.

For a genus g curve and a generic divisor of degree $n \geq g$, the number of Weierstrass points is $g(n-g+1)^2$. (We assume our base field is algebraically closed.) In the tropical case, Theorem 5 states that there are $g(n-g+1)$ Weierstrass points generically. These expressions, which differ by a factor of $(n-g+1)$, strongly suggest that the tropicalization map on Weierstrass points is generically $(n-g+1)$-to-1. Our results, in a more general version of Theorem 6, confirm that this is the case.

We also show that there are strong topological constraints on the position of the tropical Weierstrass locus.

Corollary 7 ([AGR23]). *Suppose Γ is a tropical curve of genus $g \geq 2$. Then every cycle in Γ contains a Weierstrass point in $W(K)$.*

2.3. **Tropical Manin–Mumford conjecture.** By analogy with Mordell's conjecture on finiteness of rational points, Manin and Mumford conjectured that an algebraic curve of genus 2 or more has finitely many torsion points. Given an algebraic curve with fixed basepoint x_0, we say x is a *torsion point* if $n(x-x_0)$ is linearly equivalent to the zero divisor for some positive n. Equivalently, x is a torsion point if the Abel–Jacobi embedding (with respect to x_0) sends x to the torsion subgroup $\mathrm{Jac}(X)_{\rm tors}$ of the Jacobian. This conjecture on torsion points was proved by Raynaud [Ray83]. The following stronger bound was open until very recently.

Problem 1 (Uniform bound on torsion points). Is there a constant $N(g)$ such that any algebraic curve X of genus $g \geq 2$ has $\#(X \cap \mathrm{Jac}(X)_{\rm tors}) \leq N(g)$?

This long-standing open problem was recently resolved in the affirmative by [LSW21], [Küh21; DGH21]. It remains to be seen whether tropical methods can be used to provide an additional proof.

For a metric graph Γ, there is an analogous Jacobian [MZ08] which is compatible with the Jacobian of an algebraic curve under taking the skeletons of non-Archimedean varieties [BR15]. In [Ric23], I study the tropical version of the Manin–Mumford conjecture, which asks: when a curve is embedded in its Jacobian, how often does the curve pass through torsion points of the Jacobian? We give a complete answer in the tropical case.

Theorem 8 ([Ric23]). *Let Γ be a metric graph of genus $g \geq 2$. If $\#(\Gamma \cap \mathrm{Jac}(\Gamma)_{\rm tors})$ is finite, then $\#(\Gamma \cap \mathrm{Jac}(\Gamma)_{\rm tors}) \leq 3g-3$.*

The bound of $3g-3$ in Theorem 8 answers a tropical analogue of Problem 1. However, the Manin–Mumford conjecture fails for tropical curves: for any metric graph with integer edge lengths, there are infinitely many torsion points with respect to any basepoint, no matter how large the genus is. On the other hand, I show that a metric graph does satisfy the Manin–Mumford condition if we impose certain additional constraints.

Theorem 9 ([Ric23]). *Let G be a biconnected graph of genus $g \geq 2$. For a very general choice of edge lengths $\ell : E(G) \to \mathbb{R}_{>0}$, the metric graph $\Gamma = (G, \ell)$ has $\#(\Gamma \cap \mathrm{Jac}(\Gamma)_{\mathrm{tors}}) \leq g + 1$.*

A graph is *biconnected* if it cannot be separated into two parts by cutting out one vertex; a *very general* choice of edge lengths means we exclude countably many families of positive codimension in the parameter space of edge-lengths.

There are natural higher-dimensional analogues of the Manin–Mumford condition, where we embed the d-th symmetric power of a curve, or metric graph, into its Jacobian. If D_0 is a fixed divisor of degree d, we define the d-dimensional Abel–Jacobi map

$$AJ^{(d)}_{D_0} : \mathrm{Sym}^d(\Gamma) \to \mathrm{Jac}(\Gamma) \qquad \text{by} \qquad x_1 + \cdots + x_d \mapsto [x_1 + \cdots + x_d - D_0].$$

There is a natural analogue of the Manin–Mumford condition for this embedding.

I found that a modification of the *girth* of a graph gives an upper bound on the dimension d for which the higher Manin–Mumford condition is satisfied. We call this number in 10 the *independent girth* of G; note that the girth is $\min_{C \subset E(G)} \{\#C\}$.

Theorem 10 ([Ric23]). *Suppose $\Gamma = (G, \ell)$ is a graph with very general edge lengths. Then $\mathrm{Sym}^d(\Gamma) \cap \mathrm{Jac}(\Gamma)_{\mathrm{tors}}$ is finite if and only if*

$$d \leq \min_{C \subset E(G)} \{\mathrm{rank}_{\mathcal{M}^\perp(G)}(C)\},$$

where the minimum is taken over all cycles of G and $\mathcal{M}^\perp(G)$ denotes the cographic matroid of G.

2.4. Future research objectives.

2.4.1. *Moduli space.* Conditions on canonical Weierstrass points would naturally cut out subsets in the moduli space of pointed tropical curves.

Problem 2. How is the moduli space of pointed tropical curves stratified according to conditions on Weierstrass points?

Similar stratifications for the moduli space of algebraic curves are studied by Eisenbud and Harris [EH87a], and recently by Pflueger [Pfl18]. The moduli space of algebraic curves is compatible under tropicalization with a natural moduli space of tropical curves [ACP15]. As a particular case motivated by [FJP23]: the class of the Weierstrass divisor on $\overline{\mathcal{M}}_3$ is $[\overline{\mathcal{W}}_3] = -\lambda + \psi - 3\delta_1 - 6\delta_2$ [Cuk89; EH87b]. Can we prove this relation tropically?

3. Number theory

3.1. Floor quotients.

In multiplicative number theory, one studies the prime factorization of numbers by considering the numbers as a partially ordered set (poset) under division. From this poset, we have the Möbius function $\mu(n)$, which depends on the prime factorization structure of n. It is a classical result that the Riemann Hypothesis is equivalent to the bound

$$\sum_{n \leq x} \mu(n) = O(x^{1/2+\epsilon}) \quad \text{as } x \to \infty \text{, for any } \epsilon > 0.$$

In joint work with J. Lagarias, we attempt to gain a better understanding of the classical Möbius function by defining a deformation of the partial order of natural numbers under divisibility. Using the floor function, we define the *floor quotients* of n as the numbers of the form $\{d : d = \lfloor n/k \rfloor \text{ for } k = 1, 2, \ldots, n\}$. Surprisingly, this defines a partial order relation on the whole numbers. This observation was implicit in [Car10]. In Figure 2, we show the floor quotients of $n = 16$ and their poset structure.

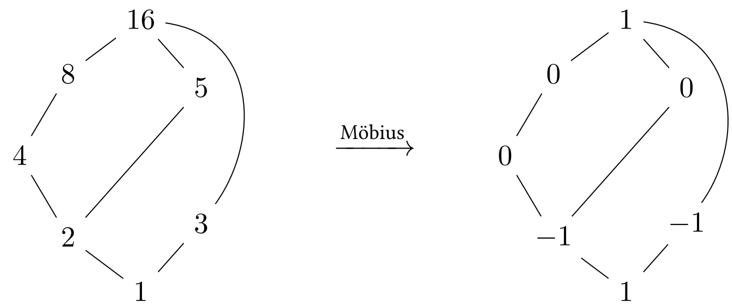


FIGURE 2. Poset of floor quotients of 16, with Möbius values $\mu_{AD}(1, d)$ on right.

In [LR24], we investigate the structure of the poset of floor quotients, answering questions in parallel with usual results in multiplicative number theory. We obtain polynomial bounds on the Möbius function of the almost-divisor poset. (The results in [Car10] suggest that $\mu_{AD}(n)$ should be considered an analogue of the sum $\sum_{n/2<i\leq n} \mu(i)$ of the usual Möbius function, rather than an analogue of $\mu(n)$.)

Theorem 11 ([LR24])**.** *Let $\mu_{AD}(n)$ denote the Möbius function of the almost-divisor poset interval from 1 to n. There is some constant $C > 0$ such that*

$$|\mu_{AD}(n)| < Cn^{1.729} \qquad \text{for all } n.$$

We believe the bound in Theorem 11 is not optimal. We hope that further investigation will yield improved bounds on the Möbius function.

Problem 3. For any $\epsilon > 0$, is there some constant $C(\epsilon)$ such that

$$|\mu_{AD}(n)| < C(\epsilon)\, n^{1+\epsilon} \qquad \text{for all } n?$$

In [LR23], we generalize the floor quotient partial order to an infinite family, parametrized by a positive integer a, such that the limit $a \to \infty$ recovers the usual multiplicative structure of positive integers.

3.2. **Rounding functions.** Discretization is the process of sending a continuous input to a discrete output, which is fundamental in many applications such as computer imaging, digital communication, and finance. *Rounding functions* are a form of regularly-spaced discretization. These create interesting behavior in the context of elementary algebra and number theory. In [LMR16; LR19; LR20] we study commutators of dilated floor functions under different scales. [Ric24a] studies the self-similar structure of Farey staircases, which are constructed from taking cumulative averages of rescaled floor functions.

3.3. p**-adic continuity and combinatorial sequences.** The p-adic topology on the integers provides a way of viewing the usually-discrete integers, or functions on integers, in an almost-continuous way. In joint work with A. O'Desky [OR23], we generalize the observation that counting derangements gives a p-adically continuous function to a larger class of derangement-like counting problems. As an application, certain classes of the counts can be combined to form a two-variable p-adic incomplete gamma function.

3.4. **Counting and geometry.** In [AAR23], joint with D. Aulicino and J. Athreya, we study a generalization of the classical problem of counting closed geodesics of bounded length in a torus.

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