Harry Richman
Summer 2015

Analysis Qualifying Review, solutions
Morning session

1. Let $E \subset \mathbb{R}^1$. Show that the characteristic function $\chi_E(x)$ is the limit of a sequence of continuous functions if and only if E is both F_σ and G_δ.

Proof. ($\Rightarrow$) First, suppose $\chi_E(x)$ is the (pointwise) limit of a sequence $\{f_n\}$ of continuous functions. Let U_n be the open set $\{x : f_n(x) > 1/2\} \subset \mathbb{R}$, and let V_n be the closed set $\{x : f_n(x) \geq 1/2\}$. Then define

$$\tilde{U}_n = \bigcup_{i \geq n} U_i = \{x : f_i(x) > 1/2 \text{ for some } i \geq n\},$$
$$\tilde{V}_n = \bigcap_{i \geq n} V_i = \{x : f_i(x) \geq 1/2 \text{ for all } i \geq n\}.$$

Note that each $\tilde{U}_n$ (resp. $\tilde{V}_n$) is open (closed) because it is a union of open sets (intersection of closed sets). We claim that $E = \cap_n \tilde{U}_n$, which shows that E is G_δ. Indeed,

$$\begin{aligned}\bigcap_{n \geq 1} \tilde{U}_n &= \bigcap_{n \geq 1} \{x : f_i(x) > 1/2 \text{ for some } i \geq n\} \\ &= \{x : \limsup_{n \to \infty} f_n(x) > 1/2\} \\ &= \{x : \chi_E(x) > 1/2\} = E\end{aligned}$$

by our assumption that $f_n \to \chi_E$ pointwise. We claim also that $E = \cup_n \tilde{V}_n$, which shows E is F_σ. Indeed,

$$\begin{aligned}\bigcup_{n \geq 1} \tilde{V}_n &= \bigcup_{n \geq 1} \{x : f_i(x) \geq 1/2 \text{ for all } i \geq n\} \\ &= \{x : \liminf_{n \to \infty} f_n(x) \geq 1/2\} \\ &= \{x : \chi_E(x) \geq 1/2\} = E.\end{aligned}$$

($\Leftarrow$) Now suppose that E is both F_σ and G_δ. Let V_n be a sequence of closed sets such that $E = \cup_n V_n$ and let U_n be a sequence of open sets such that $E = \cap_n U_n$. Without loss of generality, we may assume that the V_n are increasing, i.e. $V_n \subset V_{n+1}$, by replacing the sequence with $\tilde{V}_n = \cup_{i=1}^n V_i$. Similarly, we may assume that $U_n \supset U_{n+1}$. For each n, we have

$$V_n \subset E \subset U_n.$$

We claim that there exists a continuous function f_n, for each n, such that

$$f_n(x) = \begin{cases} 1 & \text{if } x \in V_n, \\ 0 & \text{if } x \notin U_n. \end{cases}$$

This follows from Urysohn's lemma, since V_n and U_n^c are disjoint closed subsets of $\mathbb{R}$, and $\mathbb{R}$ is a metric space and thus normal (i.e. disjoint closed sets are separated by disjoint open neighborhoods). The sequence $\{f_n\}$ converges pointwise to χ_E, as desired. $\square$

2. Let $\{g_n\}$ be a sequence of measurable functions on $[a,b]$, satisfying
   (a) $|g_n(x)| \le M$, a.e. $x \in [a,b]$;
   (b) for every $c \in [a,b]$, $\lim_{n\to\infty} \int_a^c g_n(x)dx = 0$.

   Show that for any $f \in L^1[a,b]$,

$$\lim_{n\to\infty} \int_a^b f(x)g_n(x)dx = 0.$$

Proof. We first observe that condition (b) implies that for any interval $[c_1, c_2] \subset [a,b]$, we have $\lim_{n\to\infty} \int_{c_1}^{c_2} g_n(x)dx = 0$ since

$$\begin{aligned}\lim_{n\to\infty} \left| \int_{c_1}^{c_2} g_n(x)dx \right| &= \lim_{n\to\infty} \left| \int_a^{c_2} g_n(x)dx - \int_a^{c_1} g_n(x)dx \right| \\ &\le \lim_{n\to\infty} \left| \int_a^{c_2} g_n(x)dx \right| + \lim_{n\to\infty} \left| \int_a^{c_1} g_n(x)dx \right| = 0.\end{aligned}$$

We next claim that for any measurable set $E \subset [a,b]$, $\lim_{n\to\infty} \int_E g_n(x)dx = 0$. Indeed, the previous observation implies this is true for any finite union of intervals. For any $\epsilon > 0$, there is some finite union of intervals $\tilde{E} \supset E$ such that $\mu(\tilde{E} - E) < \epsilon$, so

$$\begin{aligned}\lim_{n\to\infty} \left| \int_E g_n(x)dx \right| &= \lim_{n\to\infty} \left| \int_{\tilde{E}} g_n(x)dx - \int_{\tilde{E}-E} g_n(x)dx \right| \\ &\le \lim_{n\to\infty} \left| \int_{\tilde{E}} g_n(x)dx \right| + \lim_{n\to\infty} \int_{\tilde{E}-E} |g_n(x)|dx \le M\epsilon\end{aligned}$$

so as we let $\epsilon \to 0$ we see that the limit must be 0.

Now we show that the desired equality holds for any simple function $\tilde{f} \in L^1[a,b]$. Indeed, if $\tilde{f} = \sum_{k=1}^m a_k \chi_{E_k}$ for some measureable sets $E_k \subset [a,b]$ and $a_k \in \mathbb{R}$ then

$$\begin{aligned}\lim_{n\to\infty} \left| \int_a^b \tilde{f}(x)g_n(x)dx \right| &= \lim_{n\to\infty} \left| \int_a^b \left(\sum_{k=1}^m a_k \chi_{E_k}(x) \right) g_n(x)dx \right| = \lim_{n\to\infty} \left| \sum_{k=1}^m a_k \int_{E_k} g_n(x)dx \right| \\ &\le \lim_{n\to\infty} \sum_{k=1}^m \left| a_k \int_{E_k} g_n(x)dx \right| = \sum_{k=1}^m |a_k| \lim_{n\to\infty} \left| \int_{E_k} g_n(x)dx \right| = 0.\end{aligned}$$

Finally, since simple functions are dense in $L^1[a,b]$, for any $\delta > 0$ there is a simple function $\tilde{f}$ such that $\|f - \tilde{f}\|_1 < \delta$. Then

$$\begin{aligned}\lim_{n\to\infty} \left| \int_a^b f(x)g_n(x)dx \right| &= \lim_{n\to\infty} \left| \int_a^b \tilde{f}(x)g_n(x)dx + \int_a^b (f-\tilde{f})(x)g_n(x)dx \right| \\ &\le \lim_{n\to\infty} \left| \int_a^b \tilde{f}(x)g_n(x)dx \right| + \lim_{n\to\infty} \int_a^b |(f-\tilde{f})(x)| \cdot |g_n(x)|dx \\ &\le 0 + M \int_a^b |(f-\tilde{f})(x)|dx < M\delta,\end{aligned}$$

so the claim follows from letting $\delta \to 0$. □

3. Let $f_k(x)$, $k = 1, 2, \dots$ be increasing functions on $[a, b]$. Assume

$$\sum_{k=1}^{\infty} f_k(x)$$

is convergent on $[a, b]$. Show that

$$\left(\sum_{k=1}^{\infty} f_k(x)\right)' = \sum_{k=1}^{\infty} f_k'(x), \qquad \text{a.e. } x \in [a, b].$$

Proof. Let $F(x) = \sum_{k=1}^{\infty} f_k(x)$, and let $T_n(x) = \sum_{k=n}^{\infty} f_k(x)$ denote the tail of this summation, so that

$$F(x) = \sum_{k=1}^{n-1} f_k(x) + T_n(x)$$

for any $n \geq 1$. Taking derivatives, we have

$$F'(x) = \sum_{k=1}^{n-1} f_k'(x) + T_n'(x),$$

so it suffices to show that as $n \to \infty$, $T_n'(x) \to 0$ for a.e. x.

Note that since each f_k is increasing all the derivatives f_k', F', T_n' are non-negative. Thus for fixed x, the sequence $\{T_n'(x)\}_n$ is monotonically decreasing and bounded below by 0, so for each x, $\lim_{n\to\infty} T_n'(x) = \liminf_{n\to\infty} T_n'(x)$ exists. (We include the possibility that this limit is $+\infty$.) By Fatou's lemma,

$$\int_a^b (\liminf_{n\to\infty} T_n'(x))dx \leq \liminf_{n\to\infty} \int_a^b T_n'(x)dx,$$

and by the fundamental theorem of calculus, since T_n is increasing we have

$$\int_a^b T_n'(x)dx \leq T_n(b) - T_n(a).$$

We are given that the infinite summation defining $F(x)$ converges, so for all $x \in [a, b]$ we have $T_n(x) \to 0$ as $n \to \infty$. Thus

$$\int_a^b (\lim_{n\to\infty} T_n'(x))dx \leq \liminf_{n\to\infty} \int_a^b T_n'(x)dx \leq \liminf_{n\to\infty} (T_n(b) - T_n(a)) = 0,$$

and since the first function is non-negative this implies $\lim_{n\to\infty} T_n'(x) \equiv 0$ a.e., as desired. □

4. (a) Assume that $f \in L^\infty(\mathbb{R})$, and f is continuous at 0. Show that

$$\lim_{n\to\infty} \int \frac{n}{\pi(1 + (nx)^2)} f(x)dx = f(0).$$

(b) Assume that $f \in L^\infty(\mathbb{R})$. Show that

$$\lim_{n\to\infty} \int \frac{n}{\pi(1+n^2(x-y)^2)} f(y)dy = f(x) \qquad \text{a.e. } x \in \mathbb{R}.$$

(Hint: $\int \frac{1}{\pi(1+y^2)}dy = 1$.)

Proof. (a) By the change of variables $u = nx$,

$$\lim_{n\to\infty} \int \frac{n}{\pi(1+(nx)^2)} f(x)dx = \lim_{n\to\infty} \int \frac{f(u/n)}{\pi(1+u^2)}du,$$

so it suffices to show that the right-hand limit is equal to $f(0)$. Since $f \in L^\infty(\mathbb{R})$, there is some $M < \infty$ such that $|f(x)| < M$ a.e., and since f is continuous at 0, for any $\epsilon > 0$ there is some δ such that $|f(x) - f(0)| < \epsilon$ for any $-\delta < x < \delta$. (In particular, this implies $|f(0)| < M$.) Following the hint, we have $\int \frac{f(0)}{\pi(1+y^2)}dy = f(0)$ so

$$\begin{aligned} \left| f(0) - \int \frac{f(u/n)}{\pi(1+u^2)}du \right| &= \left| \int \frac{f(0)-f(u/n)}{\pi(1+u^2)}du \right| \\ &\le \int_{|u|<n\delta} \frac{|f(0)-f(u/n)|}{\pi(1+u^2)}du + \int_{|u|\ge n\delta} \frac{|f(0)-f(u/n)|}{\pi(1+u^2)}du \\ &\le \int_{|u|<n\delta} \frac{\epsilon}{\pi(1+u^2)}du + \int_{|u|\ge n\delta} \frac{2M}{\pi(1+u^2)}du \\ &\le \epsilon + 2M \int_{|u|\ge n\delta} \frac{1}{\pi(1+u^2)}du, \end{aligned}$$

which holds for arbitrary n. As $n \to \infty$, it is clear that the last integral approaches 0, so we have

$$\lim_{n\to\infty} \left| f(0) - \int \frac{f(u/n)}{\pi(1+u^2)}du \right| = \left| f(0) - \lim_{n\to\infty} \int \frac{f(u/n)}{\pi(1+u^2)}du \right| \le \epsilon.$$

Since this holds for arbitrary ϵ, this shows $f(0) = \lim_{n\to\infty} \int \frac{f(u/n)}{\pi(1+u^2)}du$ as desired.

(b) Since f is measurable, by Luzin's theorem for any $\epsilon > 0$ there is a continuous function $\tilde{f}$ and a closed set $E \subset \mathbb{R}$ such that $f(x) = \tilde{f}(x)$ for all $x \in E$ and $\mu(E^c) < \epsilon$. Since $|f(x)| < M$ a.e. we may also choose $\tilde{f}$ to have this same bound. Then for any x

$$\lim_{n\to\infty} \int \frac{n}{\pi(1+n^2(x-y)^2)} \tilde{f}(y)dy = \tilde{f}(x)$$

by continuity of $\tilde{f}$ (using the same argument as in part (a)), so for any $x \in E$,

$$\left| f(x) - \lim_{n\to\infty} \int \frac{n}{\pi(1+n^2(x-y)^2)} f(y)dy \right| \le \lim_{n\to\infty} \int \frac{n}{\pi(1+n^2(x-y)^2)} |\tilde{f}(y) - f(y)|dy$$

$$= \lim_{n\to\infty}\left(\int_E \frac{n|\tilde f(y)-f(y)|}{\pi(1+n^2(x-y)^2)}dy + \int_{E^c}\frac{n|\tilde f(y)-f(y)|}{\pi(1+n^2(x-y)^2)}dy\right)$$
$$\le \lim_{n\to\infty}\left(0+2M\int_{E^c}\frac{n}{\pi(1+n^2(x-y)^2)}dy\right)$$

If we choose $x \in \mathrm{int}(E)$, the interior of E, so E contains the open interval $(x-\delta, x+\delta)$ for some $\delta > 0$, then

$$\lim_{n\to\infty}\int_{E^c}\frac{n}{\pi(1+n^2(x-y)^2)}dy \le \lim_{n\to\infty}\int_{|x-y|>\delta}\frac{n}{\pi(1+n^2(x-y)^2)}dy = 0,$$

so $f(x) = \lim_{n\to\infty}\int \frac{n}{\pi(1+n^2(x-y)^2)}f(y)dy$ for any $x \in \mathrm{int}(E)$. Then the measure of the points in $\mathbb{R}$ where this equality does not hold is bounded above by $\mu(\overline{E^c}) = \mu(E^c) < \epsilon$, and since ϵ was arbitrary this equality is true almost everywhere.

□

5. Let $\{f_n\}$ be a sequence of functions in $L^p(\mathbb{R}^n)$, $1 < p < \infty$, which converge almost everywhere to a function $f \in L^p(\mathbb{R}^n)$, and suppose that there is a constant M such that $\|f_n\|_p \le M$ for all n. Show that for every $g \in L^q(\mathbb{R}^n)$, q the conjugate of p,

$$\int fg = \lim_{n\to\infty}\int f_n g.$$

Is the statement true for $p = 1$?

(Hint: you may want to use Egorov's theorem.)

Proof. We first show the statement is false for $p = 1$. Indeed, consider the functions f_n on $\mathbb{R}$ defined by

$$f_n(x) = \begin{cases} 1/n & \text{if } 0 < x < n \\ 0 & \text{otherwise.}\end{cases}$$

Then $\|f_n\|_1 = 1$ for all n, and the sequence $\{f_n\}$ converges pointwise to the zero function $f = 0$. Taking the constant function $g = 1 \in L^\infty(\mathbb{R})$, we have

$$0 = \int fg \ne \lim_{n\to\infty}\int f_n g = 1.$$

Now we prove the claim for $1 < p < \infty$. The proof relies on the fact that for any $h \in L^1(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} h = \lim_{R\to\infty}\int_{B_R} h \quad\Leftrightarrow\quad \lim_{R\to\infty}\int_{\mathbb{R}^n - B_R} h = 0$$

where $B_R \subset \mathbb{R}^n$ denotes the ball of radius R around the origin in $\mathbb{R}^n$, and

$$\lim_{\epsilon\to 0}\int_{E_\epsilon} h = 0$$

where E_ϵ is a measureable set whose measure is bounded by ϵ.

By Egorov's theorem, for any $\epsilon > 0$ (and $R > 0$ fixed) there is a subset $E_\epsilon \subset B_R$ such that $\mu(E_\epsilon) \leq \epsilon$ and the convergence $f_n \to f$ is uniform on $B_R - E_\epsilon$. Then

$$\begin{aligned}\left| \int fg - \int f_n g \right| &\leq \int |f - f_n||g| \\ &= \int_{B_R - E_\epsilon} |f - f_n||g| + \int_{E_\epsilon} |f - f_n||g| + \int_{\mathbb{R}^n - B_R} |f - f_n||g|.\end{aligned}$$

We may use Holder's inequality to replace the second and third integrals above with expressions independent of n, namely

$$\begin{aligned}\int_A |f - f_n||g| \leq \|f - f_n\|_p \|g\|_q &= \left(\int_A |f - f_n|^p \right)^{1/p} \left(\int_A |g|^q \right)^{1/q} \\ &\leq (\|f\|_p + \|f_n\|_p)\|g\|_q \leq (\|f\|_p + M)\left(\int_A |g|^q \right)^{1/q},\end{aligned}$$

and as $n \to \infty$ the first integral (on $B_R - E_\epsilon$) goes to zero by uniform convergence on a finite measure space. Thus

$$\lim_{n\to\infty} \left| \int fg - \int f_n g \right| \leq (\|f\|_p + M)\left(\int_{E_\epsilon} |g|^q \right)^{1/q} + (\|f\|_p + M)\left(\int_{\mathbb{R}^n - B_R} |g|^q \right)^{1/q}.$$

This holds for arbitrary R, ϵ so taking $\epsilon \to 0$ and $R \to \infty$, we have (since $h = |g|^q \in L^1$)

$$\lim_{n\to\infty} \left| \int fg - \int f_n g \right| = 0,$$

so $\int fg = \lim_{n\to\infty} \int f_n g$ as desired. □

Analysis Qualifying Review, solutions
Afternoon session

1. Construct an explicit analytic bijection from

$$\{z \in \mathbb{C} : |z| > 1, z \text{ not real and positive}\}$$

to

$$\{z \in \mathbb{C} : \text{Re } z > 0\}.$$

(You may write your mapping as a composition of simpler explicit mappings.)

Proof. Take $f(z) = (-iz) \circ \frac{1}{2}(z + 1/z) \circ (\sqrt{z}) = \dfrac{1}{2i}(z^{1/2} + z^{-1/2})$, where we choose the branch of the square root that is positive on positive real numbers. □

2. Let $A = \{z \in \mathbb{C} : 5 \le |z| \le 10\}$.

   (a) Prove or disprove: there is a function f analytic on a neighborhood of A and satisfying $|f(z)| < 1$ for $|z| = 10$, $|f(x)| > 1000$ for $|z| = 5$.

   (b) Prove or disprove: there is a function f analytic on a neighborhood of A and satisfying Re $f(z) < 1$ for $|z| = 10$, Re $f(z) > 1000$ for $|z| = 5$.

Proof. (a) We show such an f exists: consider $f(z) = (9/z)^n$ for some positive integer n. On $|z| = 10$, $|f(z)| = (9/10)^n < 1$, and on $|z| = 5$, $|f(z)| = (9/5)^n$ will be larger than 1000 for sufficiently large n (e.g. $n \ge 18$).

(b) We claim no such f exists. Indeed, since A is connected its image $f(A)$ will be a connected subset of $\mathbb{C}$. Let $L \subset \mathbb{C}$ denote the complex numbers with real part $= 500$. Since A is compact, its image $f(A)$ is also compact and thus closed. This implies the intersection $L \cap f(A)$ is closed. By the open mapping theorem, the interior $\text{int}(A)$ must be sent by f to an open set $f(\text{int}(A)) \subset \text{int}(f(A))$. Thus $L \cap f(\text{int}(A))$ is open in the induced subspace topology of L. We identify two possible cases: either $L \cap f(\text{int}(A)) \subsetneq L \cap f(A)$, or $L \cap f(\text{int}(A)) = L \cap f(A)$.

In the first case,

$$L \cap f(\partial A) = L \cap f(A - \text{int}(A)) \supset (L \cap f(A)) - (L \cap f(\text{int}(A)))$$

is non-empty so some point z in the boundary of A (so $|z| = 5$ or 10) must be sent to $f(z)$ with real part $= 500$, contradicting the given hypotheses for f.

In the second case, the intersection $L \cap f(\text{int}(A)) = L \cap f(A)$ is both open and closed so it is either empty or the entire line L. It cannot be the entire line because $f(A)$ is compact so it must be empty. Then since $f(A)$ is connected, it must lie entirely on one side of L (e.g. Re $f(z) < 500$) or on the other (Re $f(z) > 500$), which also contradicts the hypotheses.

□

3. For f analytic on $\mathbb{D}$ let

$$\sigma(f) = \sup\{\#f^{-1}(w) : w \in \mathbb{C}\}.$$

(a) Prove or disprove: there exists a sequence f_n of functions analytic on $\mathbb{D}$ converging uniformly on compact sets of $\mathbb{D}$ to a limit function f with all $\sigma(f_n) = 3$ but $\sigma(f) = 4$.

(b) Prove or disprove: there exists a sequence f_n of functions analytic on $\mathbb{D}$ converging uniformly on compact sets of $\mathbb{D}$ to a limit function f with all $\sigma(f_n) = 4$ but $\sigma(f) = 3$.

Proof. (a) We claim such a sequence does not exist. Indeed, suppose the sequence f_n converges to f as specified, such that $\sigma(f) = 4$. We may assume without loss of generality that f has four zeros, by adding the appropriate constant to all f, f_n. Let $z_1, \ldots, z_4 \in \mathbb{D}$ denote the four (distinct) zeros of f, each with some positive multiplicity. For each $i = 1, \ldots, 4$ choose some closed contour $\gamma_i \subset \mathbb{D}$ that encloses z_i and no other zeros, and let K_i denote the compact region enclosed by this contour. Let $\epsilon_i = \min\{|f(z)| : z \in \gamma_i\} > 0$ (since f has no zeros on γ_i). Note that $f - f_n \to 0$ uniformly on the compact sets K_i, so for sufficiently large n, $|f(z) - f_n(z)| < \epsilon_i$ for all $z \in \gamma_i$. This implies f and f_n have the same number of zeros (with multiplicty) in each K_i by Rouche's theorem. Thus $\sigma(f_n) \geq 4$ for sufficiently large n, contradicting the hypothesis.

(b) We show that such a sequence exists as follows. Let $\phi(z) = \dfrac{z+i}{iz+1}$ be a Möbius transformation sending the unit disk to the upper half-plane; then set $f_n(z) = \phi(z)^{6+1/n}$ (choosing the principal branch of the logarithm), so the sequence f_n converges uniformly on compacts sets to the limit function $f(z) = \phi(z)^6$. It is straightforward to verify that for any $\alpha > 0$,

$$\#(\phi^\alpha)^{-1}(w) = \begin{cases} \lceil \alpha/2 \rceil & \text{if } 0 < \text{Arg } w < \{\pi\alpha\} \\ \lceil \alpha/2 \rceil - 1 & \text{if } \{\pi\alpha\} \leq \text{Arg } w \leq 2\pi \end{cases}$$

where $\phi^\alpha(z) := \phi(z)^\alpha$ and we define $\{\pi\alpha\}$ to be the unique angle equivalent to $\pi\alpha$ in the range $(0, 2\pi]$, so $\sigma(\phi^\alpha) = \lceil \alpha/2 \rceil$. Thus $\sigma(f_n) = \lceil 3 + \frac{1}{2n} \rceil = 4$, while $\sigma(f) = 3$ as desired. □

4. Let $f \colon \mathbb{D} \to \mathbb{D}$ be an analytic function defined on a neighborhood of $\overline{\mathbb{D}}$ and satisfying

- $f(\overline{\mathbb{D}}) \subset \mathbb{D}$;
- $f(0) = 0$.

Let $f^{\circ n} = f \circ \cdots \circ f$, n times. Show that $f^{\circ n}(z) \to 0$ for $z \in \mathbb{D}$.

Proof. Consider the function $g(z) = f(z)/z$. Since $f(0) = 0$, g is analytic on $\overline{\mathbb{D}}$, and on the boundary $\partial\overline{\mathbb{D}} = \{z : |z| = 1\}$, the magnitudes of $f(z)$ and $g(z)$ coincide. Thus

$$\max_{|z|=1}\{|g(z)|\} = \max_{|z|=1}\{|f(z)/z|\} = \max_{|z|=1}\{|f(z)|\}.$$

Call this maximum λ (which exists by compactness of $\partial\overline{\mathbb{D}}$); since $f(\overline{\mathbb{D}}) \subset \mathbb{D}$ we have $0 \leq \lambda < 1$. By the maximum modulus principle,

$$|g(z)| = |f(z)/z| \leq \lambda \quad \text{for all } z \in \overline{\mathbb{D}}.$$

But this is equivalent to $|f(z)| \leq \lambda|z|$ for all $z \in \overline{\mathbb{D}}$, and by induction this implies

$$|f^{\circ n}(z)| \leq \lambda^n |z| \to 0 \quad \text{as } n \to \infty, \quad (z \in \overline{\mathbb{D}} \text{ fixed})$$

so $f^{\circ n}(z) \to 0$ as desired. □

5. Suppose $\{f_n\}$ is a uniformly bounded sequence of analytic functions on a domain Ω such that $\{f_n(z)\}$ converges for every $z \in \Omega$.

   (a) Show that the convergence is uniform on every compact subset of Ω.

   (b) Must $\{f_n'\}$ converge uniformly on every compact subset of Ω? Prove or disprove.

Proof. (a) Fix a compact subset $K \subset \Omega$. Suppose for a contradiction that the convergence $f_n \to f$ is not uniform on K. Then for some $\epsilon > 0$, there are infinitely many f_n such that

$$\sup_{z \in K} |f_n(z) - f(z)| > \epsilon.$$

Let $\{f_{n_k}\}$ consist of all such functions. Then $\{f_{n_k}\}$ is also uniformly bounded, so by Montel's theorem there is a subsequence $f_{n_{k_j}}$ that converges uniformly to some function g analytic on K, i.e.

$$\lim_{j \to \infty} \left(\sup_{z \in K} |f_{n_{k_j}}(z) - g(z)| \right) = 0.$$

Then $f_{n_{k_j}} \to g$ pointwise, and since $f_{n_{k_j}}$ is also a subsequence of $\{f_n\}$, which converges pointwise to f, we must have $g = f$. But this contradicts our assumption that

$$\sup_{z \in K} |f_{n_{k_j}}(z) - g(z)| = \sup_{z \in K} |f_{n_{k_j}}(z) - f(z)| > \epsilon$$

for all j, so we must have uniform convergence on K as desired.

(b) Yes; $\{f_n'\}$ must converge uniformly on compact subsets as well. Indeed, for a fixed compact subset $K \subset \Omega$ we may choose an open neighborhood $K \subset \Omega_1 \subset \Omega$ such that Ω_1 has compact closure; let K_1 denote this closure. Then take another open neighborhood $K_1 \subset \Omega_2 \subset \Omega$ with compact closure K_2. (We may assume Ω_1 and Ω_2 are connected.) Then the distance $|z - w|$ for $z \in \Omega_1$, $w \in \partial K_2$ is bounded below by some constant $\epsilon > 0$. Moreover, the hypotheses on f_n gives a uniform bound $|f_n(z)| < M$ for all $z \in K_2$ and all n (where M may depend on K_2). By Cauchy's integral formula,

$$f_n'(z) = \frac{1}{2\pi i} \oint_{\partial K_2} \frac{f_n(\xi)}{(\xi - z)^2} d\xi$$

for any $z \in \Omega_1$, so taking magnitudes

$$|f_n'(z)| \leq \frac{1}{2\pi} \oint_{\partial K_2} \frac{|f_n(\xi)|}{|\xi - z|^2} d\xi \leq \frac{1}{2\pi} |\partial K_2| \frac{M}{\epsilon^2} =: M'$$

which shows that $\{f_n'\}$ is uniformly bounded on Ω_1. Also by Cauchy's formula,

$$\begin{aligned} |f_n'(z) - f'(z)| &= \left| \frac{1}{2\pi i} \oint_{\partial K_2} \frac{f_n(\xi) - f(\xi)}{(\xi - z)^2} d\xi \right| \\ &\leq \frac{1}{2\pi} |\partial K_2| \frac{\sup_{K_2} |f_n(\xi) - f(\xi)|}{\epsilon^2} \to 0 \quad \text{as } n \to \infty \end{aligned}$$

by uniform convergence of $f_n \to f$, so $f_n' \to f'$ pointwise on Ω_1. Thus we have uniform convergence of f_n' on K by the argument in (a).

$\square$