Analysis Qualifying Review, solutions
Morning session

1. Let f be a non-negative measurable function on $(0,1)$. Assume that there is a constant c, such that
$$\int_0^1 (f(x))^n dx = c, \quad n = 1, 2, \dots.$$
Show that there is a measurable set $E \subset (0,1)$ such that
$$f(x) = \chi_E(x), \quad \text{for a.e. } x \in (0,1).$$

Proof. First we claim that $f \leq 1$ almost everywhere. Indeed, if $m(x \colon f(x) > 1) > 0$ then we could choose some $\epsilon > 0$ such that $m(x \colon f(x) \geq 1 + \epsilon) =: M > 0$, and as $n \to \infty$
$$\int_0^1 (f(x))^n dx \geq \int_0^1 (1+\epsilon)^n \chi_{\{f(x) \geq 1+\epsilon\}} dx = M(1+\epsilon)^n \to \infty$$
which contradicts the hypothesis that this integral is constant as n varies.

Now define $E = \{x \colon f(x) = 1\}$ which is measurable by assumption that f is measurable. We claim that $f = \chi_E$ a.e., and $c = m(E)$. Since $f \leq 1$ a.e., the sequence $\{f^n\}$ is monotonically decreasing and $f^n \to \chi_E$ pointwise a.e. By dominated convergence,
$$\lim_{n\to\infty} \int (f(x))^n dx = \int \chi_E(x) dx = m(E)$$
but by hypothesis this limit is also equal to $\lim_{n\to\infty} c = c$, so $c = m(E)$. By construction the function $f(x) - \chi_E(x)$ is non-negative, and
$$\int (f(x) - \chi_E(x)) dx = c - m(E) = 0$$
so $f = \chi_E$ a.e., as desired. □

2. Let f be locally integrable on $\mathbb{R}^n$, $1 < p < \infty$. Show that the following are equivalent:

   (i) $f \in L^p(\mathbb{R}^n)$;

   (ii) there exists $M > 0$ such that for any finite collection of mutually disjoint measurable sets $E_1, E_2, \dots, E_k$ with $0 < m(E_i) < \infty$ for $1 \leq i \leq k$,
$$\sum_{i=1}^k \left(\frac{1}{m(E_i)} \right)^{p-1} \left| \int_{E_i} f(x) dx \right|^p \leq M.$$

Proof. (i)⇒(ii): If $f \in L^p(\mathbb{R}^n)$, then on any E with finite measure we may apply Holder's inequality to bound
$$\begin{aligned} \left| \int_E f(x) dx \right| \leq \int_E |f(x)| dx &\leq \left(\int_E |f|^p dx \right)^{1/p} \left(\int_E 1^q dx \right)^{1/q} \\ &= (m(E))^{1/q} \left(\int_E |f|^p dx \right)^{1/p} \end{aligned}$$

where q is the conjugate of p, so in particular $p/q = p-1$. Then

$$\left|\int_E f(x)dx\right|^p \leq m(E)^{p-1}\left(\int_E |f|^p dx\right),$$

so applying this for each E_i we have

$$\sum_{i=1}^k \left(\frac{1}{m(E_i)}\right)^{p-1}\left|\int_{E_i} f(x)dx\right|^p \leq \sum_{i=1}^k \left(\int_{E_i} |f(x)|^p dx\right) \leq \|f\|_p^p,$$

which is finite. Thus we may take $M = \|f\|_p^p$.

(ii)$\Rightarrow$(i): We consider instead the contrapositive; if $f \notin L^p$, we claim that no finite bound M exists in (ii). Suppose g_n is a monotone increasing sequence of non-negative simple functions that converges pointwise to $|f|$. By monotone convergence,

$$\lim_{n\to\infty}\int g_n^p dx = \int |f|^p dx = \infty.$$

But since g_n is simple, say

$$g_n(x) = \sum_{i=1}^{k_n} c_{i,n}\chi_{E_{i,n}}(x), \quad c_{i,n} \in \mathbb{R}_{>0}$$

with $E_{i,n}$ disjoint for $i = 1, \ldots, k_n$, it is straightfoward to verify that

$$\sum_{i=1}^{k_n} \left(\frac{1}{m(E_{i,n})}\right)^{p-1}\left|\int_{E_{i,n}} g_n(x)dx\right|^p = \sum_{i=1}^k m(E_{i,n})c_i^p = \int g_n^p dx.$$

Under a suitable (finite) refinement of $\{E_{i,n}\}_i$ we may assume that the values of f are either all positive or all negative on each $E_{i,n}$, so that

$$\sum_{i=1}^{k_n} \left(\frac{1}{m(E_{i,n})}\right)^{p-1}\left|\int_{E_{i,n}} f(x)dx\right|^p = \sum_{i=1}^{k_n} \left(\frac{1}{m(E_{i,n})}\right)^{p-1}\left(\int_{E_{i,n}} |f(x)|dx\right)^p \geq \int g_n^p dx$$

by assumption that $|f| \geq g_n$. As $n \to \infty$, the integral on the right grows without bound, so (ii) fails as claimed. $\square$

3. Let $f\colon \mathbb{R}\times[0,1] \to \mathbb{R}$ be a measurable function such that for any $y \in [0,1]$,

$$\int_{\mathbb{R}} f^2(x,y)dx \leq 1.$$

Prove that there exists a sequence $x_n \to +\infty$ such that

$$\int_0^1 f(x_n, y)dx \to 0.$$

Proof. Integrating the given inequality over y, we have

$$\int_0^1 \left(\int_{\mathbb{R}} f^2(x,y)dx \right) dy \le \int_0^1 1 dy = 1,$$

and as the integrand is non-negative, we may apply Tonelli's theorem to interchange the order of integration:

$$\int_{\mathbb{R}} \left(\int_0^1 f^2(x,y)dy \right) dx \le 1.$$

Let $x_n \in [n, n+1]$ be chosen such that

$$\int_0^1 f^2(x_n,y)dy \le \int_n^{n+1} \left(\int_0^1 f^2(x,y)dy \right) dx,$$

(i.e. the y-integral at x_n is no larger than average on $[n, n+1]$) so

$$\sum_{n=1}^{\infty} \int_0^1 f^2(x_n,y)dy \le \int_{\mathbb{R}} \left(\int_0^1 f^2(x,y)dy \right) dx \le 1.$$

The convergence of this sum implies that $\int_0^1 f^2(x_n,y)dy \to 0$. To get the desired statement on $\int_0^1 f(x_n,y)dy$, we apply the Cauchy-Schwarz inequality:

$$\left| \int_0^1 f(x_n,y)dy \right| \le \left(\int_0^1 f^2(x_n,y)dy \right)^{1/2} \left(\int_0^1 1^2 dy \right)^{1/2} = \sqrt{\int_0^1 f^2(x_n,y)dy}.$$

It is clear by continuity of the square root function that $\sqrt{a_n} \to 0$ whenever $a_n \to 0$, so the above inequality shows that $\int_0^1 f(x_n,y)dy \to 0$ as desired. $\square$

4. Let $E_k \subset [a,b]$ be measurable sets, $k \in \mathbb{N}$, and there exists $\delta > 0$ such that $m(E_k) \ge \delta$ for all k. Assume that $a_k \in \mathbb{R}$ satisfy

$$\sum_{k=1}^{\infty} |a_k| \chi_{E_k}(x) < \infty \quad \text{for a.e. } x \in [a,b].$$

Show that $\displaystyle\sum_{k=1}^{\infty} |a_k| < \infty$.

Proof. Let $S(x) = \sum_{k=1}^{\infty} |a_k|\chi_{E_k}(x)$, let $S_n(x) = \sum_{k=1}^{n} |a_k|\chi_{E_k}(x)$ denote the partial sums, and let $T_n(x) = \sum_{k=n+1}^{\infty} |a_k|\chi_{E_k}(x)$ denote the tail ends of this sum. Let F denote the set where these sums are finite:

$$F = \{x : S(x) < \infty\} = \{x : T_n(x) < \infty\} \quad \text{for any } n.$$

By definition $S_n \to S$ pointwise on F, and by hypothesis we have

$$m([a,b] - F) = 0.$$

If this convergence were uniform on some subset $G \subset F$, then

$$\sup_{x\in G} |S(x) - S_n(x)| = \sup_{x\in G} |T_n(x)| \to 0 \quad \text{as } n \to \infty,$$

so in particular $\sup_G |T_n(x)|$ would be finite for sufficiently large n, and thus for all n. Since $m(G) < \infty$ this implies T_n is integrable on G, and it follows that S is integrable as well:

$$\int_G S(x)dx = \int_G S_n(x)dx + \int_G T_n(x)dx < \infty.$$

But from the definition of S as a linear combination of characteristic functions,

$$\int_G S(x)dx = \sum_{k=1}^{\infty} |a_k| m(E_k \cap G) \geq \inf_k \{m(E_k \cap G)\} \sum_k |a_k|,$$

so to show that the sum $\sum |a_k|$ is finite, it suffices to find a G (where $S_n \to S$ uniformly) such that $m(E_k \cap G)$ is bounded below by a positive constant. But this is a direct consequence of Egorov's theorem: for any $\epsilon > 0$, we have uniform convergence on some G with $m(F - G) < \epsilon$, so $m(G^c) = m([a, b] - G) = m([a, b] - F) + m(F - G) < \epsilon$ implies

$$m(E_k \cap G) = m(E_k - G^c) \geq m(E_k) - m(G^c) \geq \delta - \epsilon.$$

Choosing $\epsilon < \delta$, we have the desired result. □

5. Let $A, B \subset \mathbb{R}^d$. Assume $A \cup B$ is measurable, and $m(A \cup B) < \infty$. If

$$m(A \cup B) = m^*(A) + m^*(B),$$

show that A and B are measurable.

Proof. We first show that there is a measurable set U containing A such that $m^*(A) = m(U)$. Indeed, for any $n \in \mathbb{N}$ there is a measurable $U_n \supset A$ such that $m(U_n) \leq m^*(A) + 1/n$ by definition of outer measure. Then the intersection $U = \cap_n U_n$ has the desired properties.

Similarly, let V be a measurable set containing B such that $m^*(B) = m(V)$. Then

$$m(U) + m(V) \geq m(U \cup V) \geq m(A \cup B) = m^*(A) + m^*(B) = m(U) + m(V)$$

so we have the equality $m(U \cup V) = m(U) + m(V)$. This implies $m(U \cap V) = 0$, so the subset $A \cap V \subset U \cap V$ must also be measurable with measure 0. Observe that

$$A = (A - V) \cup (A \cap V)$$

so it suffices to show that the set $A - V$ is measurable. But $A - V = (A \cup B) - V$ since V contains B, so as the difference of two measure sets it is measurable as well. Thus A is measurable, and by symmetry B is also measurable. □

Analysis Qualifying Review, solutions
Afternoon session

1. Is there a function f, analytic at the origin, taking values $f(\frac{1}{2n}) = f(\frac{1}{2n-1}) = \frac{1}{2n}$, for $n \in \mathbb{N}$?

   Proof. No, such a function does not exist. If $f(1/2n) = 1/2n$ for all $n \in \mathbb{N}$, then the zeros of the function $f(z) - z$ would have an accumulation point at the origin, which implies that $f(z) - z = 0$ identically since it is analytic at the origin. Then $f(z) = z$, so we cannot have $f(\frac{1}{2n-1}) = 1/2n$ as desired. □

2. Find and classify the singularities of the function $f(z) = \sin\left(\frac{1}{\sin(z)}\right)$.

   Proof. Recall that $\sin(z)$ is an entire function with simple zeros at $k\pi$ for all $k \in \mathbb{Z}$, and an essential singularity at ∞. Therefore the function $\frac{1}{\sin(z)}$ has simple poles at all $k\pi$ and a non-isolated singularity at ∞. Now consider the composite function $f(z) = \sin\left(\frac{1}{\sin(z)}\right)$. At any complex number $z \neq k\pi$, the inside argument $\frac{1}{\sin(z)}$ is a non-zero complex number so f has no singularity at z. As $z \to 0$, the inside argument tends to infinity ($\frac{1}{\sin(z)} = \frac{1}{z} + O(z) \to \infty$ as $z \to 0$), so f has an essential singularity at 0. Since f satisfies the periodic relation

   $$f(z+\pi) = \sin\left(\frac{1}{\sin(z+\pi)}\right) = \sin\left(\frac{1}{-\sin(z)}\right) = -f(z)$$

   the same argument shows that f has an essential singularity at each $k\pi$. □

3. Let f be an entire function such that $f(z) \in \mathbb{R}$ whenever $z \in \mathbb{R}$. Assume that $|f(z)| \leq 1$ on the boundary of the rectangle with the vertices $-1, 1, 1+i, -1+i$. Prove that $|f'(\frac{i}{10})| \leq \frac{10}{9}$.

   Proof. By the maximum modulus principle, the bound $|f(z)| \leq 1$ holds for all z inside the given rectangle, which we denote R_1. By Schwarz reflection, the values of f on the reflection $\overline{R_1}$ of R_1 across the real axis are given by $f(\bar{z}) = \overline{f(z)}$. In particular, the modulus is the same after reflection, $|f(\bar{z})| = |f(z)|$, so the bound $|f(z)| \leq 1$ holds in the "doubled" rectangle $R_2 = R_1 \cup \overline{R+1}$ with vertices at $\{\pm 1 \pm i\}$.

   This shows that the restriction of f to the closed unit disk $\mathbb{D} \subset R_2$ must have image inside $\mathbb{D}$, so we may apply the Schwarz-Pick bound

   $$\left| f'\left(\frac{i}{10}\right) \right| \leq \frac{1 - |f(i/10)|^2}{1 - |i/10|^2} \leq \frac{1}{1 - 1/100} = \frac{100}{99}$$

   This in particular implies the weaker bound $|f'(i/10)| \leq \frac{10}{9}$. □

4. Suppose Ω is a bounded region, f is holomorphic on Ω and

   $$\limsup_{n\to\infty} |f(z_n)| \leq M$$

   for every sequence z_n in Ω which converges to a boundary point of Ω. Prove that $|f(z)| \leq M$ for all $z \in \Omega$.

Proof. Recall that every region $\Omega \subset \mathbb{C}$ admits an *exhaustion* $\{K_n\}$, namely a nested sequence of compact sets $K_1 \subset K_2 \subset \cdots \subset \Omega$ whose union is all of Ω. In this case, where Ω is bounded, we may define K_n as

$$K_n = \{x \in \Omega : d(x, \partial\Omega) \geq \frac{1}{n}\}.$$

On each K_n define

$$M_n = \sup_{z \in K_n} |f(z)|.$$

By compactness this supremum is achieved for some $z_n \in K_n$, and by the maximum modulus principle z_n can be chosen to lie on the boundary of K_n. (It must lie on the boundary unless f is (locally) constant.) Since Ω is bounded, the sequence $\{z_n\}$ must have some convergent subsequence, which we denote $\{z_{n_k}\}$. This subsequence must converge to a point in the closure of Ω, but it cannot converge to a point in the interior because any interior point is also in the interior of K_n for sufficiently large n. Thus $\{z_{n_k}\}$ converges to a boundary point of Ω, so

$$\limsup_{k\to\infty} |f(z_{n_k})| = \limsup_{k\to\infty} M_{n_k} \leq M$$

by hypothesis. But the sequence $M_1 \leq M_2 \leq \cdots$ in non-decreasing since the K_n are nested, so $\sup_n\{M_n\} \leq M$ which implies that $|f(z)| \leq M$ for any $z \in \cup_n K_n = \Omega$ as desired. ☐

5. Let $f \in C^1(\mathbb{R})$ be a function with compact support.

   (a) Show that for any $x \in \mathbb{R}$,

   $$\text{p.v.} \int_{-\infty}^{\infty} \frac{f(\zeta)}{\zeta - x} d\zeta := \lim_{\epsilon \to 0} \int_{|\zeta - x| > \epsilon} \frac{f(\zeta)}{\zeta - x} d\zeta \quad \text{exists.}$$

   (b) Prove that

   $$\lim_{y\to 0^+} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(\zeta)}{\zeta - (x+iy)} d\zeta = \frac{1}{2} f(x) + \frac{1}{2\pi i} \cdot \text{p.v.} \int_{-\infty}^{\infty} \frac{f(\zeta)}{\zeta - x} d\zeta.$$

Proof. (a) Observe that

$$\int_{|\zeta-x|>\epsilon} \frac{f(\zeta)}{\zeta - x} d\zeta = \int_{|\zeta-x|\geq 1} \frac{f(\zeta)}{\zeta - x} d\zeta + \int_{1>|\zeta-x|>\epsilon} \frac{f(\zeta) - f(x)}{\zeta - x} d\zeta + \int_{1>|\zeta-x|>\epsilon} \frac{f(x)}{\zeta - x} d\zeta$$

The first integral converges since the f has compact support, and the third integral vanishes for any ϵ by symmetry of $\frac{1}{\zeta - x}$. Thus it suffices to show that the second integral approaches a limit as $\epsilon \to 0$; we claim in fact

$$\lim_{\epsilon\to 0} \int_{1>|\zeta-x|>\epsilon} \frac{f(\zeta) - f(x)}{\zeta - x} d\zeta = \int_{x-1}^{x-1} \frac{f(\zeta) - f(x)}{\zeta - x} d\zeta \quad \text{exists.}$$

By assumption f is differentiable at x, so the limit

$$f'(x) = \lim_{\zeta \to x} \frac{f(\zeta) - f(x)}{\zeta - x} \quad \text{exists.}$$

Thus for any $\delta_1 > 0$ there is a $\delta_2 > 0$ such that $|f'(x) - \frac{f(\zeta)-f(x)}{\zeta-x}| < \delta_1$ for any ζ satisfying $|\zeta - x| < \delta_2$. In particular, $|\frac{f(\zeta)-f(x)}{\zeta-x}| < |f'(x)| + \delta_1$ must be bounded on the interval $\zeta \in (x-\delta_2, x+\delta_2)$, and it is bounded by continuity on the compact set $\zeta \in [x-1, x-\delta_2] \cup [x+\delta_2, x+1]$.

(b) Observe

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{f(\zeta)}{\zeta-(x+iy)}d\zeta &= \int_{-\infty}^{\infty} \frac{f(\zeta)(\zeta-x+iy)}{(\zeta-x)^2+y^2}d\zeta \\ &= \int_{-\infty}^{\infty} \frac{f(\zeta)(\zeta-x)}{(\zeta-x)^2+y^2}d\zeta + iy\int_{-\infty}^{\infty} \frac{f(\zeta)}{(\zeta-x)^2+y^2}d\zeta\end{aligned}$$

It is clear by dominated convergence that for any fixed $\epsilon > 0$,

$$\lim_{y\to 0^+} \int_{|\zeta-x|>\epsilon} \frac{f(\zeta)(\zeta-x)}{(\zeta-x)^2+y^2}d\zeta = \int_{|\zeta-x|>\epsilon} \frac{f(\zeta)}{\zeta-x}d\zeta.$$

To get bounds on the remaining part, let $\delta_\epsilon = \sup_{|h|\le\epsilon} \left|\frac{f(x+h)-f(x)}{h} - f'(x)\right|$ so that $|f(x+h) - f(x) - f'(x)h| \le \delta_\epsilon |h|$ for all $|h| < \epsilon$:

$$\begin{aligned}\lim_{y\to 0^+} \left|\int_{|\zeta-x|\le\epsilon} \frac{f(\zeta)(\zeta-x)}{(\zeta-x)^2+y^2}d\zeta\right| &= \lim_{y\to 0^+} \left|\int_{|\zeta-x|\le\epsilon} \frac{f(x)(\zeta-x)}{(\zeta-x)^2+y^2}d\zeta + \int_{|\zeta-x|\le\epsilon} \frac{(f(\zeta)-f(x))(\zeta-x)}{(\zeta-x)^2+y^2}d\zeta\right| \\ &\le \lim_{y\to 0^+} \left|f(x)\int_{|\zeta-x|\le\epsilon} \frac{(\zeta-x)}{(\zeta-x)^2+y^2}d\zeta\right| \\ &\quad + \lim_{y\to 0^+} \left|\int_{|\zeta-x|\le\epsilon} \frac{f'(x)(\zeta-x)^2}{(\zeta-x)^2+y^2}d\zeta\right| + \lim_{y\to 0^+} \left|\int_{|\zeta-x|\le\epsilon} \frac{\delta_\epsilon(\zeta-x)^2}{(\zeta-x)^2+y^2}d\zeta\right| \\ &= 0 + 2\epsilon f'(x) + 2\epsilon\delta_\epsilon\end{aligned}$$

by symmetry of the first integral and by bounded convergence of the second and third. As $\epsilon \to 0$, it is clear that δ_ϵ decreases to 0 by definition of the derivative (since $f \in C^1$). Taking the above two results together, as $\epsilon \to 0$, we have

$$\lim_{y\to 0^+} \int_{-\infty}^{\infty} \frac{f(\zeta)(\zeta-x)}{(\zeta-x)^2+y^2}d\zeta = \lim_{\epsilon\to 0}\left(\int_{|\zeta-x|>\epsilon} \frac{f(\zeta)}{\zeta-x}d\zeta + O(\epsilon)\right) = \text{p.v.}\int_{-\infty}^{\infty} \frac{f(\zeta)}{\zeta-x}d\zeta.$$

Now consider the remaining term:

$$\begin{aligned}\lim_{y\to 0^+} iy\int_{-\infty}^{\infty} \frac{f(\zeta)}{(\zeta-x)^2+y^2}d\zeta &= \lim_{y\to 0^+} \frac{i}{y}\int_{-\infty}^{\infty} \frac{f(\zeta)}{(\frac{\zeta-x}{y})^2+1}d\zeta \\ &= \lim_{y\to 0^+} i\int_{-\infty}^{\infty} \frac{f(x+y\xi)}{\xi^2+1}d\xi \\ &= i\int_{-\infty}^{\infty} \frac{f(x)}{\xi^2+1}d\xi = i\pi f(x).\end{aligned}$$

The claim follows.

(To justify the last line in the above equation, we may check for any $R > 0$ that

$$\begin{aligned}
\lim_{y\to 0^+}\left|\int_{-R}^{R}\frac{f(x+y\xi)-f(x)}{\xi^2+1}d\xi\right| &\le \lim_{y\to 0^+}\int_{-R}^{R}\frac{|f(x+y\xi)-f(x)|}{\xi^2+1}d\xi \\
&\le \lim_{y\to 0^+}\left(\sup_{|\delta|<yR}|f(x+\delta)-f(x)|\right)\int_{-R}^{R}\frac{1}{\xi^2+1}d\xi \\
&= 0
\end{aligned}$$

by continuity of f. It also follows by dominated convergence under $\frac{\sup|f|}{\xi^2+1}$.)

$\square$