Morning session

1. Construct a measurable subset A of $(0,1)$ such that $m(A) < 1$ and $m(A \cap (a,b)) > 0$ for any $(a,b) \subset (0,1)$.

   Proof. Let $\{q_n\} = \{q_1, q_2, \ldots\}$ be an enumeration of the rational numbers in the interval $(0,1)$, and let E_n be the open interval

$$E_n = (q_n - 5^{-n}, q_n + 5^{-n}) \cap (0,1).$$

   Then $A = \cup_n E_n \subset (0,1)$ is measurable, as a countable union of measurable sets, and

$$m(A) \leq \sum_{n \geq 1} m(E_n) \leq \sum_{n \geq 1} 2 \cdot 5^{-n} = 1/2.$$

   Finally any open interval $(a,b) \subset (0,1)$ contains some rational q_n, so the intersection $E_n \cap (a,b)$ has positive measure and thus $m(A \cap (a,b))$ is positive as well. Thus A satisfies the specified conditions. □

2. Let $\{f_k(x)\}$ be a sequence of non-negative measurable functions on E and $m(E) < \infty$. Show that $\{f_k(x)\}$ converges in measure to 0 if and only if

$$\lim_{k \to \infty} \int_E \frac{f_k(x)}{1 + f_k(x)} dx = 0.$$

   Proof. ($\Rightarrow$): Suppose $f_k \to 0$ in measure, meaning that for any $\epsilon > 0$

$$m(x : |f_k(x)| \geq \epsilon) \to 0 \quad \text{as} \quad k \to \infty.$$

   Then since $y \mapsto \frac{y}{1+y} : \mathbb{R}_{\geq 0} \to [0,1)$ is monotonically increasing,

$$\begin{aligned}
\int_E \frac{f_k(x)}{1 + f_k(x)} dx &= \int_{|f_k| < \epsilon} \frac{f_k(x)}{1 + f_k(x)} dx + \int_{|f_k| \geq \epsilon} \frac{f_k(x)}{1 + f_k(x)} dx \\
&\leq \int_{|f_k| < \epsilon} \frac{\epsilon}{1 + \epsilon} dx + \int_{|f_k| \geq \epsilon} 1 dx \\
&\leq m(E) \frac{\epsilon}{1 + \epsilon} + m(|f_k| \geq \epsilon).
\end{aligned}$$

   Taking $k \to \infty$, we see that $\lim_{k \to \infty} \int_E \frac{f_k(x)}{1 + f_k(x)} dx \leq m(E) \frac{\epsilon}{1 + \epsilon}$, and as $\epsilon \to 0$ this bound goes to zero, so the limit is zero as claimed.

   ($\Leftarrow$): Observe that for any $\epsilon > 0$,

$$\begin{aligned}
\int_E \frac{f_k(x)}{1 + f_k(x)} dx &= \int_{|f_k| \geq \epsilon} \frac{f_k(x)}{1 + f_k(x)} dx + \int_{|f_k| < \epsilon} \frac{f_k(x)}{1 + f_k(x)} dx \\
&\geq \int_{|f_k| \geq \epsilon} \frac{\epsilon}{1 + \epsilon} dx \\
&= m(|f_k| \geq \epsilon) \frac{\epsilon}{1 + \epsilon} \geq 0.
\end{aligned}$$

If the given sequence of integrals converges to zero, then (ignoring the non-zero constant factor $\frac{\epsilon}{1+\epsilon}$)

$$\lim_{k\to\infty} m(x : |f_k(x)| \geq \epsilon) = 0$$

as well. Thus $f_k \to 0$ in measure. □

3. Let $1 \leq p < \infty$, $f \in L^p(\mathbb{R}^n)$. Let

$$f_*(\lambda) = m(\{x \colon |f(x)| > \lambda\}), \quad \lambda > 0.$$

Show that

(a) $p\int_0^\infty \lambda^{p-1} f_*(\lambda) d\lambda = \int |f(x)|^p dx$

(b) $\lim\limits_{\lambda\to\infty} \lambda^p f_*(\lambda) = 0, \quad \lim\limits_{\lambda\to 0} \lambda^p f_*(\lambda) = 0.$

Proof. (a) Suppose $g = \sum_k c_k \chi_{E_k}$ is a simple function in $L^p(\mathbb{R}^n)$, where $\{E_k\}$ are disjoint. Then

$$g_*(\lambda) = \sum_{\substack{k \text{ s.t.} \\ |c_k|>\lambda}} m(E_k) = \sum_k m(E_k)\chi_{[0,|c_k|)}(\lambda),$$

and

$$\begin{aligned}
\int |g|^p dx = \sum_k m(E_k)|c_k|^p &= \sum_k m(E_k)\left(\int_0^{|c_k|} p\lambda^{p-1} d\lambda\right) \\
&= \sum_k m(E_k)\left(\int_0^\infty p\lambda^{p-1}\chi_{[0,|c_k|)} d\lambda\right) \\
&= \int_0^\infty p\lambda^{p-1}\left(\sum_k m(E_k)\chi_{[0,|c_k|)}\right) d\lambda = \int_0^\infty p\lambda^{p-1} g_*(\lambda) d\lambda.
\end{aligned}$$

Thus the desired equality holds for simple functions. For an arbitrary $f \in L^p(\mathbb{R}^n)$, we may find a monotonically increasing sequence of simple functions g_n that converge pointwise to $|f|$, so that $\|g_n\|_p \to \|f\|_p$ by monotone convergence. Under these conditions $g_n \leq f$,

$$\{x : |g_n(x)| > \lambda\} \subset \{x : |f(x)| > \lambda\} \quad \Rightarrow \quad (g_n)_*(\lambda) \leq f_*(\lambda)$$

for all λ, so the sequence $\lambda^{p-1}(g_n)_*$ will also converge monotonically to $\lambda^{p-1} f_*$. By monotone convergence in $L^1(\mathbb{R})$, $\|p\lambda^{p-1}(g_n)_*\|_1 \to \|p\lambda^{p-1} f_*\|_1$. Thus

$$\int |f(x)|^p dx = \lim_{n\to\infty} \int |g_n(x)|^p dx = \lim_{n\to\infty} p\int_0^\infty \lambda^{p-1}(g_n)_*(\lambda) d\lambda = p\int_0^\infty \lambda^{p-1} f_*(\lambda) d\lambda$$

as desired.

(b) Given a sequence $\lambda_n \to 0$ of positive numbers, define $g_n = \lambda_n \cdot \chi_{\{|f|>\lambda_n\}}$. It is clear that g_n is non-negative, bounded above by $|f|$, and converges pointwise (in fact, uniformly) to 0. Thus by dominated convergence

$$\lim_{n\to\infty} \int (g_n)^p dx = 0.$$

But

$$\lambda_n^p f_*(\lambda_n) = \int \lambda_n^p \cdot \chi_{\{|f|>\lambda_n\}} dx = \int (g_n)^p dx$$

so the above convergence of integrals implies $\lim_{n\to\infty} \lambda_n^p f_*(\lambda_n) = 0$. Since this holds for any $\lambda_n \to 0$, we may conclude that $\lim\limits_{\lambda\to 0} \lambda^p f_*(\lambda) = 0$.

To see that $\lim\limits_{\lambda\to\infty} \lambda^p f_*(\lambda) = 0$, simply repeat the above argument with $\lambda_n \to \infty$. (The convergence $g_n \to 0$ is no longer uniform, but it is still dominated.)

□

4. Let $K = \{f\colon (0,+\infty) \to \mathbb{R} | \int_0^\infty f^4(x)dx \le 1\}$. Evaluate

$$\sup_{f\in K} \int_0^\infty f^3(x)e^{-x}dx.$$

Proof. By Holder's inequality, for $p = 4/3$ and $q = 4$,

$$\begin{aligned}\int_0^\infty |f^3(x)e^{-x}|dx &\le \left(\int_0^\infty |f^3(x)|^{4/3}dx\right)^{3/4}\left(\int_0^\infty e^{-4x}dx\right)^{1/4} \\ &= \left(\int_0^\infty |f^4(x)|dx\right)^{3/4}\left(\frac{1}{4}\right)^{1/4}.\end{aligned}$$

Thus for any $f \in K$, the above integral is bounded above by $1/\sqrt{2}$. This bound is achieved when $f(x) = \sqrt{2}e^{-x} \in K$ (i.e. when $\|f\|_4 = 1$ and $f^4 \sim e^{-4x}$ so Holder gives an equality), so

$$\sup_{f\in K} \int_0^\infty f^3(x)e^{-x}dx = \frac{1}{\sqrt{2}}.$$

□

5. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function such that $\int_{\mathbb{R}} |f(x)|dx < \infty$. Show that the sequence

$$h_n(x) = \frac{1}{n}\sum_{k=1}^n f\left(x + \frac{k}{n}\right)$$

converges in $L^1(\mathbb{R})$.

Proof. Let $h\colon \mathbb{R} \to \mathbb{R}$ denote the function $h(x) = \int_0^1 f(x+\xi)d\xi$, which exists by hypothesis that $f \in L^1(\mathbb{R})$. By Tonelli's theorem

$$\begin{aligned}\int_{\mathbb{R}} |h(x)|dx = \int_{\mathbb{R}} \left| \int_0^1 f(x+\xi)d\xi \right| dx &\le \int_{\mathbb{R}} \left(\int_0^1 |f(x+\xi)|d\xi\right) dx \\ &= \int_0^1 \left(\int_{\mathbb{R}} |f(x+\xi)|dx\right) d\xi = \|f\|_1 < \infty,\end{aligned}$$

so $h \in L^1(\mathbb{R})$. We claim that $h_n \to h$ in L^1. Indeed if f is Riemann integrable, then we have convergence pointwise by definition of the Riemann integral:

$$h(x) = \int_0^1 f(x+\xi)d\xi = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^n f\left(x + \frac{k}{n}\right) = \lim_{n\to\infty} h_n(x).$$

To show convergence in L^1, we will at first assume stronger conditions on f; namely, suppose f is continuous with bounded support. Then f is uniformly continuous, so for any $\epsilon > 0$ there is some $\delta > 0$ such that

$$|x - y| < \delta \quad \Rightarrow \quad |f(x) - f(y)| < \epsilon.$$

Then for any $n > 1/\delta$,

$$\begin{aligned}
|h(x) - h_n(x)| &= \left| \int_0^1 f(x+\xi)d\xi - \frac{1}{n}\sum_{k=1}^n f(x+k/n)\right| \\
&\le \sum_{k=1}^n \left| \int_{(k-1)/n}^{k/n} f(x+\xi)d\xi - \frac{1}{n} f(x+k/n)\right| \\
&\le \sum_{k=1}^n \int_{(k-1)/n}^{k/n} \Big| f(x+\xi) - f(x+k/n) \Big| d\xi \le \sum_{k=1}^n \int_{(k-1)/n}^{k/n} \epsilon \, d\xi = \epsilon.
\end{aligned}$$

Moreover if $\operatorname{supp}(f) \subset (-R, R)$, then $h(x) = h_n(x) = 0$ for any $x \le -R-1$ or $x \ge R$, so for n sufficiently large (depending on ϵ)

$$\|h - h_n\|_1 = \int_{\mathbb{R}} |h(x) - h_n(x)| dx = \int_{-R-1}^{R} |h(x) - h_n(x)| dx \le (2R+1)\epsilon.$$

Since ϵ was arbitrary this shows $h_n \to h$ in L^1 under the above assumptions on f.

Now for *arbitrary* $f \in L^1$, there is a sequence $f^{(m)}$ of continuous, bounded support functions that converge to f in L^1. Let $h_n^{(m)}$, $h^{(m)}$ denote the corresponding functions for $f^{(m)}$. It is straightforward to check that

$$\|h - h^{(m)}\|_1 \le \|f - f^{(m)}\|_1 \quad \text{and} \quad \|h_n - h_n^{(m)}\|_1 \le \|f - f^{(m)}\|_1$$

under these conditions by exchanging the order of integration, so

$$\begin{aligned}
\|h - h_n\|_1 &\le \|h - h^{(m)}\|_1 + \|h^{(m)} - h_n^{(m)}\|_1 + \|h_n^{(m)} - h_n\|_1 \\
&\le \|h^{(m)} - h_n^{(m)}\|_1 + 2\|f - f^{(m)}\|_1.
\end{aligned}$$

As $n \to \infty$ (with m fixed) this shows

$$\lim_{n\to\infty} \|h - h_n\|_1 \le 2\|f - f^{(m)}\|_1$$

and by assumption this upper bound goes to zero as $m \to \infty$. This proves the claim. $\square$

Afternoon session

1. Assume that 0 is an isolated singularity of an analytic function $f \neq 0$. Determine the type of the singularity if
$$\sum_{n=1}^{\infty} |f(1/n)|^{1/n} < +\infty.$$

Proof. If the given series converges, then in particular the terms $|f(1/n)|^{1/n}$ in the summation must approach 0, so for any $1 > \epsilon > 0$ we have
$$|f(1/n)|^{1/n} < \epsilon \quad \Leftrightarrow \quad |f(1/n)| < \epsilon^n$$
for n sufficiently large. This implies that for any fixed integer k,
$$\lim_{n\to\infty} \frac{|f(1/n)|}{(1/n)^k} \leq \lim_{n\to\infty} \frac{\epsilon^n}{(1/n)^k} = 0.$$
If f has either a removable singularity or a pole at 0, then f has a Taylor series expansion around 0 of the form
$$f(z) = a_k z^k + a_{k+1} z^{k+1} + \cdots ,$$
where $a_k \neq 0$. (If $k \geq 0$ then the singularity is removable; if $k < 0$ it is a pole.) Then for any sequence of points z_n approaching 0, $f(z_n)/z_n^k \to a_k$, so taking magnitudes
$$\lim_{n\to\infty} \frac{|f(z_n)|}{|z_n|^k} = |a_k| > 0.$$
Considering the sequence $z_n = 1/n \to 0$, this shows the singularity at 0 cannot be removable or a pole if the given series converges, so it must be essential.

To see that it *is* possible for the given series to converge for some f with an essential singularity at 0, we may take as an example $f(z) = e^{-1/z^2}$. ($\sum_{n=1}^{\infty} |e^{-n^2}|^{1/n} = \sum_{n=1}^{\infty} e^{-n} = \frac{1}{e-1}$.) □

2. Show that for $x, y \in \mathbb{R}$,
$$|y| \leq |\sin(x+iy)| \leq e^{|y|}.$$

Proof. Recall that $\sin(z) = \frac{1}{2i}(e^{iz} - e^{-iz})$, so
$$\sin(x+iy) = \frac{1}{2i}(e^{ix}e^{-y} - e^{-ix}e^{y}).$$
Taking magnitudes and applying the triangle quality in two ways, we have
$$|\sin(x+iy)| \leq \frac{1}{2}\Big(|e^{ix}e^{-y}| + |e^{-ix}e^{y}|\Big) = \frac{1}{2}(e^{-y} + e^{y}) \leq e^{|y|}$$
since $e^{|y|} = \max\{e^{-y}, e^{y}\}$, and
$$|\sin(x+iy)| \geq \frac{1}{2}\Big||e^{ix}e^{-y}| - |e^{-ix}e^{y}|\Big| = \frac{1}{2}|e^{-y} - e^{y}| = |\sinh(y)|$$

so, observing that $|\sinh(y)| = \sinh|y|$, it suffices to prove $|y| \leq \sinh|y|$. Starting from the easier inequality $0 \leq \sinh|y|$,

$$1 = \cosh(0) \leq \cosh(0) + \int_0^{|y|} \sinh(y')dy' = \cosh|y|,$$

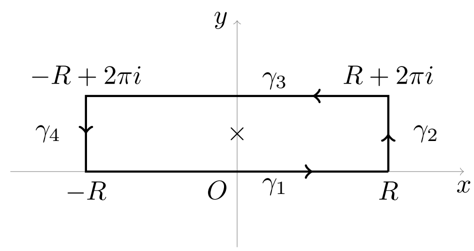
and integrating this inequality again gives us the desired result:

$$|y| = \int_0^{|y|} dy' \leq \int_0^{|y|} \cosh(y')dy' = \sinh(|y|).$$

(We in fact have the stronger bounds $\sinh|y| \leq |\sin(x+iy)| \leq \cosh(y)$.) $\square$

3. Let $a \in (0,1)$. Find
$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x} dx.$$

Proof. Consider integrating the meromorphic function $f(z) = \dfrac{e^{az}}{1+e^z}$ along the rectangular contour with endpoints $-R, R, R+2\pi i, -R+2\pi i$, with segments labeled as below:



The poles of f occur where $z = (2k+1)\pi i$ for $k \in \mathbb{Z}$, so the only pole in our contour occurs at $z = \pi i$. The residue at this pole is

$$\text{Res}_f(\pi i) = \lim_{z \to \pi i} \frac{(z - \pi i)e^{az}}{1+e^z} = \lim_{z \to \pi i} \frac{a(z-\pi i)e^{az} + e^{az}}{e^z} = -e^{a\pi i}.$$

The integral along γ_1 as $R \to \infty$ is the value we are asked to find:

$$I = \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x} dx = \lim_{R\to\infty} \int_{\gamma_1} f dz.$$

The integral along γ_3 we can relate to the integral along γ_1:

$$\int_{\gamma_3} f dz = \int_R^{-R} \frac{e^{a(x+2\pi i)}}{1+e^{x+2\pi i}} dx = -e^{2a\pi i} \int_{-R}^{R} \frac{e^{ax}}{1+e^x} dx = -e^{2a\pi i} \int_{\gamma_1} f dz,$$

and the integrals along γ_2 and γ_4 go to zero as $R \to \infty$:

$$\left| \int_{\gamma_2} f dz \right| = \left| \int_0^{2\pi} \frac{e^{a(R+iy)}}{1+e^{R+iy}} i dy \right| \leq \int_0^{2\pi} \frac{e^{aR}}{|1+e^{R+iy}|} dy \leq 2\pi \left(\frac{e^{aR}}{e^R - 1} \right)$$

and

$$\lim_{R\to\infty} \frac{e^{aR}}{e^R - 1} = \lim_{R\to\infty} \frac{1}{e^{(1-a)R} - e^{-aR}} = 0 \quad \text{since } 0 < a < 1.$$

Thus

$$\lim_{R\to\infty} \int_{\gamma_1+\gamma_2+\gamma_3+\gamma_4} f dz = (1 - e^{2a\pi i})I$$

which by Cauchy's theorem is equal to $2\pi i \sum \mathrm{Res}_f = -2\pi i e^{a\pi i}$. Thus

$$I = \frac{-2\pi i e^{a\pi i}}{1 - e^{2a\pi i}} = \frac{\pi}{\sin(a\pi)}.$$

□

4. Find a conformal mapping of domain $\overline{\mathbb{C}}\backslash\{x + xi : 1 \le x \le 2\}$ to the upper half plane. (Here $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.) It is enough to represent the mapping as a composition of several conformal mappings.

   Proof. We claim the following map works:

$$f(z) = \sqrt{-\frac{z - (1+i)}{z - (2+2i)}},$$

   where we choose the branch of the square root sending the positive real axis to itself. The expression inside the square root is a möbius transformation sending the segment $\{x+xi : 1 \le x \le 2\}$ to the positive real axis $\cup\{\infty\}$, so composing this with the square root will give us the desired behavior, since $\sqrt{z}$ sends $\mathbb{C}\backslash\{\text{positive reals}\}$ to the upper half plane conformally. □

5. Denote by $\mathbb{D}$ the unit disk: $\mathbb{D} = \{z : |z| < 1\}$. Let $\{f_k : \mathbb{D} \to \mathbb{C}\}_{k\in\mathbb{N}}$ be a normal family. Prove that the functions

$$g_k(z) = f_k(e^{ik}z), \quad k \in \mathbb{N}$$

   form a normal family.

   Proof. For $n \in \mathbb{N}$ let $K_n \subset \mathbb{D}$ denote the compact set

$$K_n = \{z : |z| \le 1 - 1/n\}.$$

   Since $\{f_k\}$ is a normal family, $|f_k(z)|$ must be bounded on each K_n uniformly as k varies. Let M_n be such a bound, so

$$|f_k(z)| < M_n \quad \text{for all } z \in K_n,\ k = 1, 2, \ldots.$$

   We claim this is also a uniform bound for $\{g_k\}$ on K_n. Indeed, for each fixed k, n, the rotation $z \mapsto e^{ik}z$ sends K_n to itself so

$$\sup_{z\in K_n} |g_k(z)| = \sup_{z\in K_n} |f_k(e^{ik}z)| = \sup_{w\in K_n} |f_k(w)| < M_n.$$

   Thus $\{g_k\}$ is a normal family by Montel's theorem, since it is uniformly bounded on compact subsets. (It is clear that any compact subset of $\mathbb{D}$ is contained in some K_n.) □