Morning session

1. Prove that there exists an analytic function $f\colon \mathbb{D} \to \mathbb{D}$ such that $f(1/2) = f(-1/2)$ and $f'(z) \neq 0$ for all $z \in \mathbb{D}$.

   Proof. Consider the map $e^{2\pi i z}\colon \mathbb{C} \to \mathbb{C}$. This map is 1-periodic and has non-vanishing derivative $2\pi i e^{2\pi i z}$, and the closed disk $\overline{\mathbb{D}}$ is sent by this map to some bounded region $\{z : |z| < M\}$. Then
   $$f(z) = \frac{1}{M} e^{2\pi i z}$$
   has the desired properties. □

2. Let f be a polynomial such that
   $$|f(z)| \leq 1 - |z|^2 + |z|^{1000}$$
   for all $z \in \mathbb{C}$. Prove that $|f(0)| \leq 0.2$.

   Proof. Consider the values of f on the circle of radius 0.9 around the origin:
   $$|z| = 0.9 \quad \Rightarrow \quad |f(z)| \leq 1 - (0.9)^2 + (0.9)^{1000} = 0.2 - 0.01 + (0.9)^{1000} < 0.2.$$
   Thus by the maximum modulus principle $|f(z)| < 0.2$ for all z in the interior of this circle (since f is entire), so in particular $|f(0)| < 0.2$. □

3. Let $f_k\colon \mathbb{D} \to \mathbb{C}$ be a normal family of analytic functions and let $h_k\colon \mathbb{D} \to \mathbb{D}$ be analytic functions satisfying $h_k(0) = 0$. Prove that the functions
   $$g_k(z) = f_k(h_k(z))$$
   form a normal family.

   Proof. For $n \in \mathbb{N}$ let $K_n \subset \mathbb{D}$ denote the compact set
   $$K_n = \{z : |z| \leq 1 - 1/n\}.$$
   Since $\{f_k\}$ is a normal family, $|f_k(z)|$ must be bounded on each K_n uniformly as k varies. Let M_n be such a bound, so
   $$|f_k(z)| < M_n \quad \text{for all } z \in K_n,\ k = 1, 2, \ldots.$$
   We claim this is also a uniform bound for $\{g_k\}$ on K_n. Indeed, by Schwarz's lemma $h_k(K_n) \subset K_n$ for all k, n so for each fixed k, n
   $$\sup_{z\in K_n} |g_k(z)| = \sup_{z\in K_n} |f_k(h_k(z))| \leq \sup_{w\in K_n} |f_k(w)| < M_n.$$
   Thus $\{g_k\}$ is a normal family by Montel's theorem. (It is clear that any compact subset of $\mathbb{D}$ is contained in some K_n.) □

4. Let
$$\Omega_1 = \mathbb{C} \backslash \Big(\{0\} \cup \{1/n : n \in \mathbb{N}\} \Big)$$
and
$$\Omega_2 = \mathbb{C} \backslash \{z : \text{Im } z = 0, \quad |\text{Re } z| \geq 1\}.$$
Construct a non-constant analytic function $f \colon \Omega_1 \to \Omega_2$ or show that this is impossible.

Proof. We claim this is not possible; namely, any analytic $f \colon \Omega_1 \to \Omega_2$ must be constant.

Suppose $f \colon \Omega_1 \to \Omega_2$ is analytic. We can post-compose with an analytic isomorphism $\Omega_2 \to \mathbb{D}$ to obtain an analytic function $\tilde{f} \colon \Omega_1 \to \mathbb{D}$. Since we have the uniform bound $|\tilde{f}(z)| < 1$ for all $z \in \Omega_1$, the behavior of $\tilde{f}$ near the isolated points $1/n$ cannot be a pole or essential singularity, so $\tilde{f}$ must have a removable singularity at each $1/n$. Thus, $\tilde{f}$ can be extended to a map $\mathbb{C} \backslash \{0\} \to \mathbb{D}$, and by the same reasoning this function must have a removable singularity at 0 so $\tilde{f}$ may be extended to an entire function $\mathbb{C} \to \mathbb{D}$. But by Liouville's theorem all such functions are constant, so f is constant, as claimed. □

5. Let $f(z)$ be the branch of $\sqrt{z^2 - 1}$ on $|z| > 1$ satisfying $\displaystyle\lim_{z \to \infty} \frac{f(z)}{z} = 1$.

   (a) Determine the coefficients $\alpha, \beta, \gamma, \delta, \epsilon$ in the Laurent expansion
   $$f(z) = \alpha z + \beta + \gamma z^{-1} + \delta z^{-2} + \epsilon z^{-3} + \cdots .$$

   (b) Compute $\displaystyle\int_{|z|=2} (5 + 6z + 7z^2) f(z) dz$.

Proof. (a) From the limit expression it is clear that $\alpha = 1$. Squaring the given Laurent expansion gives
$$\begin{aligned} f(z)^2 &= \alpha^2 z^2 + (2\alpha\beta)z + (2\alpha\gamma + \beta^2) + (2\alpha\delta + 2\beta\gamma)z^{-1} + (2\alpha\epsilon + 2\beta\delta + \gamma^2)z^{-2} + \cdots \\ &= z^2 + (2\beta)z + (2\gamma + \beta^2) + (2\delta + 2\beta\gamma)z^{-1} + (2\epsilon + 2\beta\delta + \gamma^2)z^{-2} + \cdots \end{aligned}$$
which must match coefficient-wise with the Laurent expansion $f(z)^2 = z^2 - 1$. Thus we have the system of equations
$$\begin{aligned} 2\beta &= 0 \\ 2\gamma + \beta^2 &= -1 \\ 2\delta + 2\beta\gamma &= 0 \\ 2\epsilon + 2\beta\delta + \gamma^2 &= 0 \end{aligned}$$
which has the solution $(\alpha, \beta, \gamma, \delta, \epsilon) = (1, 0, -1/2, 0, -1/8)$.

(b) Observe that the integrand $(5+6z+7z^2)f(z)$ is not meromorphic in the "interior" region $\{z : |z| \leq 2\}$, but *is* meromorphic in the region $\{z : |z| \geq 2\} \cup \{\infty\}$, with a single pole at ∞. Thus in order to evaluate the given contour integral using Cauchy's formula, we switch coordinates to a neighborhood of $z = \infty$. The above Laurent expansion shows how to express f in such a neighborhood: changing coordinates to $w = 1/z$, we may express f in a neighborhood of $w = 0$ as

$$f(w) = w^{-1} - \frac{1}{2}w - \frac{1}{8}w^3 + \cdots = w^{-1} \sum_{n \geq 0} \binom{1/2}{n} (-w^2)^n$$

which has radius of convergence $|w| < 1$. Changing the given intergral to the coordinate w, we have

$$\begin{aligned}
\int_{\substack{|z|=2 \\ \text{(ccw)}}} (5 + 6z + 7z^2) f(z) dz &= \int_{\substack{|w|=1/2 \\ \text{clockwise}}} (5 + 6w^{-1} + 7w^{-2}) f(w^{-1})(-w^{-2} dw) \\
&= \int_{\substack{|w|=1/2 \\ \text{c-clockwise}}} (7w^{-4} + 6w^{-3} + 5w^{-2})(w^{-1} - \frac{1}{2}w - \frac{1}{8}w^3 + \cdots) dw \\
&= \int_{|w|=1/2} (\cdots + (-\frac{7}{8} - \frac{5}{2})w^{-1} + \cdots) dw.
\end{aligned}$$

Since the only pole occurs at $w = 0$, with residue $-27/8$, Cauchy's integral formula tells us

$$\int_{\partial K} g(w) dw = 2\pi i \sum_{\text{poles } a \in K} \text{Res}_g(a) = -\frac{27}{4}\pi i.$$

□

Afternoon session

1. Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function. Prove that the set of points where f is differentiable is measurable.

Proof. Define

$$\underline{f}'(x) = \liminf_{h\to 0} \frac{f(x+h)-f(x)}{h}, \qquad \overline{f}'(x) = \limsup_{h\to 0} \frac{f(x+h)-f(x)}{h}.$$

It suffices to prove that both $\underline{f}'$ and $\overline{f}'$ are measurable, since then

$$\{x : f \text{ differentiable at } x\} = \{x : \overline{f}'(x) - \underline{f}'(x) = 0\}.$$

For this, observe that

$$\underline{f}'(x) = \lim_{n\to\infty} \left(\inf_{0<|h|<1/n} \frac{f(x+h)-f(x)}{h} \right) = \lim_{n\to\infty} \left(\inf_{\substack{0<|h_i|<1/n \\ h_i \text{ rational}}} \frac{f(x+h_i)-f(x)}{h_i} \right)$$

where it suffices to consider rational h_i because f is continuous. For any non-zero h it is clear that the difference quotient

$$\Delta_h f(x) = \frac{f(x+h)-f(x)}{h}$$

defines a measurable function on $[a,b] \cap [a-h, b-h]$, which we may extend to $[a,b]$ by $\pm\infty$ as appropriate. The above equation expresses $\underline{f}'$ as the limit of a countable infimum of measurable functions. Thus $\underline{f}'$ is measurable, and $\overline{f}'$ is measurable by the same reasoning. □

2. Let $f_1, f_2, \ldots, f, g$ be measurable functions on a measure space $(X, \mathcal{A}, \mu)$. Assume that $f_n \to f$ in measure and $f_n \le g$ a.e. Prove that $f \le g$ a.e.

Proof. Under the given conditions $h_n = g - f_n$ is a sequence of a.e. non-negative functions that converge in measure to $h = g - f$, and the claim is that h must also be non-negative a.e. For any $\epsilon > 0$

$$\{x : h(x) \le -\epsilon\} \subset \{x : |h_n(x) - h(x)| > \epsilon/2\} \cup \{x : h_n(x) < 0\},$$

which implies that

$$\mu(\{x : h(x) \le -\epsilon\}) \le \mu(\{|h_n(x) - h(x)| > \epsilon/2\})$$

since h_n is non-negative a.e. by assumption. This holds for any n, so it also holds for the limit as $n \to \infty$. But this limit is 0 by assumption that $h_n \to h$ in measure:

$$\begin{aligned} \mu(x : h(x) < 0) &= \lim_{\epsilon\to 0^+} \mu(x : h(x) \le -\epsilon) \\ &\le \lim_{\epsilon\to 0^+} \left(\lim_{n\to\infty} \mu(x : |h_n(x) - h(x)| > \epsilon/2) \right) \\ &= \lim_{\epsilon\to 0^+} (0) = 0. \end{aligned}$$

(*Alternate proof*: If $f_n \to f$ in measure, then there is some subsequence f_{n_k} that converges pointwise to f a.e. Since $f_{n_k} \le g$ a.e., this clearly implies $f \le g$ a.e.) □

3. Let $q_1, q_2, \ldots$ be an enumeration of $\mathbb{Q} \cap [0,1]$ and let $r, t \in (0,1)$. Consider the set

$$E := \left\{ x \in [0,1] : \sum_{j=1}^{\infty} t^j |x - q_j|^{-r} < \infty \right\}.$$

(a) Show that $E \neq [0,1] \backslash \mathbb{Q}$.

(b) Show that $m([0,1] \backslash E) = 0$.

Proof. (a) To ease notation let

$$f_j(x) = t^j |x - q_j|^{-r} \quad \text{and} \quad S(x) = \sum_{j=1}^{\infty} f_j(x).$$

It is clear that the inclusion $E \subset [0,1] \backslash \mathbb{Q}$ holds, since for any rational $q_j \in [0,1]$ there is a single term in the summation (namely f_j) that is infinite at q_j. It suffices to show that this inclusion is strict, i.e. there is some irrational $p \in [0,1]$ such that $p \notin E$.

We observe that for every index j, there is a positive-length closed interval on which $f_j(x) \geq 1/2$, and moreover we may choose this interval to lie inside an arbitrarily small neighborhood of q_j, and to *not* contain the point q_j. Let I_1 be such an interval, not containing q_1, on which $f_1(x) \geq 1/2$. Let j_2 be the smallest index such that $q_{j_2} \in I_1$, and let $I_2 \subset I_1$ be an interval not containing q_{j_2} on which $f_{j_2}(x) \geq 1/2$. Then continue in the same manner: for any $i \geq 2$ let j_i be the smallest index such that $q_{j_i} \in I_{i-1}$ and let $I_i \subset I_{i-1}$ be an interval not containing q_{j_i} on which $f_{j_i}(x) \geq 1/2$. This process gives us an increasing sequence of indices $1 = j_1 < j_2 < j_3 < \cdots$ and a nested (/decreasing) sequence of closed intervals $I_1 \supset I_2 \supset \cdots$.

Now consider the intersection $\cap_i I_i$. It is clear that this intersection is disjoint from E since

$$S(x) = \sum_{j \geq 1} f_j(x) \geq \sum_{j_i} f_{j_i}(x) \geq \sum_{j_1} 1/2 = \infty \quad \text{for any } x \in \cap_i I_i.$$

We also claim that $\cap_i I_i$ is disjoint from the rationals $\mathbb{Q}$. Indeed, if we had some rational $q_j \in \cap_i I_i$, we could compare the index j with the sequence $j_1 < j_2 < \cdots$; eventually $j_n > j$, but since both

$$q_j, q_{j_n} \in I_{n-1},$$

this contradicts our assumption that we choose the smallest index in each step above. Since the intervals I_i form a nested sequence of compact sets, their intersection is non-empty by Cantor's theorem so any $p \in \cap_i I_i$ suffices for our purposes.

(b) If $m([0,1] \backslash E) > 0$, then the integral of S over $[0,1]$ must clearly be infinite, so it suffices to show that this integral is finite:

$$\int_0^1 S(x)dx = \int_0^1 \sum_{j=1}^{\infty} t^j |x - q_j|^{-r} dx = \sum_{j=1}^{\infty} t^j \int_0^1 |x - q_j|^{-r} dx$$
$$\leq \sum_{j=1}^{\infty} t^j \int_{q_j - 1}^{q_j + 1} |x - q_j|^{-r} dx = \sum_{j=1}^{\infty} t^j \frac{2}{1-r} = \frac{2}{1-r} \frac{t}{1-t} < \infty.$$

(We may interchange summation signs since all terms are positive.) Thus the claim follows.

$\square$

4. For $f \in L^1(\mathbb{R}, m)$ let $Tf(x) = \int_{x-1}^{x+1} f dm$.

   (a) Show that if $Tf = 4f$ a.e. then $f = 0$ a.e.

   (b) Does the conclusion of (a) still hold if we only assume that f is integrable on each bounded interval in $\mathbb{R}$?

Proof. (a) We first verify that $Tf \in L^1$, applying Tonelli's theorem:

$$\int_{\mathbb{R}} |Tf(x)|dx \leq \int_{\mathbb{R}} \int_{-1}^{1} |f(x+y)|dydx = \int_{-1}^{1} \int_{\mathbb{R}} |f(x+y)|dxdy = 2\|f\|_1 < \infty.$$

But by linearity of the 1-norm $\|4f\|_1 = 4\|f\|_1$. Thus $Tf = 4f$ a.e. implies

$$\|Tf\|_1 = 4\|f\|_1 \leq 2\|f\|_1 \quad \Rightarrow \quad \|f\|_1 = 0.$$

Since the 1-norm is non-degenerate this means $f = 0$ a.e.

(b) No; consider the counter-example $f = e^{kx}$ (with k some parameter to be chosen later):

$$Tf(x) = \int_{-1}^{1} e^{k(x+y)}dy = e^{kx} \int_{-1}^{1} e^{ky}dy = e^{kx}\left(\frac{e^k - e^{-k}}{k}\right) =: A(k)f(x).$$

The function $A(k) = \frac{1}{k}(e^k - e^{-k})$ is continuous for $k > 0$, has limit $A(k) \to 2$ as $k \to 0$, and approaches $+\infty$ as $k \to \infty$. Thus some value k_0 satisfies $A(k_0) = 4$.

$\square$

5. Prove the sequence

$$f_n(x) = n^{1/2} \exp\left(-\frac{n^2x^2}{x+1}\right)$$

converges in $L^p([0, +\infty), m)$ for $1 \leq p < 2$ and diverges for $p \geq 2$.

Proof. It is clear the $f_n \to 0$ pointwise a.e. (everywhere except at $x = 0$), so we must show that $\|f_n\|_p \to 0$ as $n \to \infty$ for $1 \leq p < 2$, and $\|f_n\|_p \not\to 0$ for $p \geq 2$.

To show convergence in the desired range, we note the two inequalities

$$0 \leq \frac{x^2}{2} \leq \frac{x^2}{x+1} \quad \text{on } [0,1], \quad \text{and} \quad 0 \leq x - 1 \leq \frac{x^2}{x+1} \text{ on } [1, \infty),$$

so that

$$\begin{aligned}
\|f_n\|_p^p &= n^{p/2}\bigg(\int_0^1 \exp\bigg(-\frac{pn^2x^2}{x+1}\bigg)dx + \int_1^\infty \exp\bigg(-\frac{pn^2x^2}{x+1}\bigg)dx\bigg) \\
&\le n^{p/2}\bigg(\int_0^1 \exp\bigg(-\frac{pn^2x^2}{2}\bigg)dx + \int_1^\infty \exp\Big(-pn^2(x-1)\Big)dx\bigg) \\
&\le n^{p/2}\bigg(\int_0^\infty \exp(-y^2)\frac{dy}{\sqrt{pn^2/2}} + \int_0^\infty \exp(-y)\frac{dy}{pn^2}\bigg) \\
&= n^{p/2}\bigg(\frac{1}{n\sqrt{p/2}}\frac{\sqrt{\pi}}{2} + \frac{1}{pn^2}\bigg) = C_1\frac{n^{p/2}}{n} + C_2\frac{n^{p/2}}{n^2}
\end{aligned}$$

where C_1 and C_2 are positive constants depending on p. If $p < 2$ then both terms in this upper bound go to 0 as $n \to \infty$, so $\|f_n\|_p \to 0$ as claimed.

To show divergence in the desired range, we use that $0 \le \frac{x^2}{x+1} \le x^2$ on $[0,\infty)$ to see

$$\begin{aligned}
\|f_n\|_p^p &= n^{p/2}\bigg(\int_0^\infty \exp\bigg(-\frac{pn^2x^2}{x+1}\bigg)dx\bigg) \ge n^{p/2}\bigg(\int_0^\infty \exp\Big(-pn^2x^2\Big)dx\bigg) \\
&= n^{p/2}\bigg(\int_0^\infty \exp(-y^2)\frac{dy}{\sqrt{pn^2}}\bigg) = n^{p/2}\bigg(\frac{1}{n\sqrt{p}}\frac{\sqrt{\pi}}{2}\bigg) = C\frac{n^{p/2}}{n}
\end{aligned}$$

where C is some positive constant depending on p. If $p \ge 2$ then this lower bound is $\ge C$ so $\|f_n\|_p \not\to 0$ as $n \to \infty$. □