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Morning session

1. Prove or disprove: If E is an open subset of $\mathbb{R}$ with $m(E) = 1$ then there is a finite union of intervals F containing E with $m(F) < 1.1$.

   Proof. Consider the countable union of open intervals $E = \cup_{n\geq 1}(n, n+1/2^n) \subset \mathbb{R}$. It is clear that $m(E) = \sum_{n\geq 1} 1/2^n = 1$. However, if F is a finite union of intervals with $m(F) < \infty$, then F is bounded as a subset of $\mathbb{R}$, so F *cannot* contain E. □

2. Let $f \in L_1 \cap L_4$ (on some measure space). Prove that the function

$$\begin{aligned} [1,4] &\to \mathbb{R} \\ p &\mapsto \|f\|_p \end{aligned}$$

   is continuous.

   Proof. Let $N\colon [1,4] \to \mathbb{R}$ denote the given "norm" function. It suffices to prove $\tilde{N}(p) = \|f\|_p^p = \int |f|^p$ is continuous, since then we have $N(p) = \exp(\log(\tilde{N}(p))/p)$ is the composition of continuous functions. Let

$$E = \{x : |f(x)| \leq 1\} \quad \text{and} \quad F = \{x : |f(x)| > 1\}.$$

   We first check that $\|f\|_p^p < \infty$ for all $p \in [1,4]$, so that $\tilde{N}$ is well-defined. Indeed

$$\int |f|^p = \int_E |f|^p + \int_F |f|^p \leq \int_E |f| + \int_F |f|^4 \leq \|f\|_1 + \|f\|_4^4 < \infty.$$

   Now to show continuity of $\tilde{N}$ at $p \in [1,4]$, we must prove $\tilde{N}(p+\epsilon_n) \to \tilde{N}(p)$ for any sequence $\epsilon_n \to 0$ (for which $p + \epsilon_n \in [1,4]$). Given such a sequence, define

$$g_n = |f|^{p+\epsilon_n} - |f|^p \quad \Rightarrow \quad \|g_n\|_1 \geq |\tilde{N}(p+\epsilon_n) - \tilde{N}(p)|.$$

   It is clear that $g_n \to 0$ pointwise, and what wish to prove is that $g_n \to 0$ in L_1. As $\int |g_n| = \int_E |g_n| + \int_F |g_n|$, it suffices to check this on E and F separately. On E, the sequence g_n is dominated by $2|f|$:

$$|g_n(x)| \leq |f(x)|^{p+\epsilon_n} + |f(x)|^p \leq 2|f(x)| \quad \text{for any } x \in E,$$

   while of F, g_n is dominated by $2|f|^4$. Thus the limit of integrals goes to zero as claimed, by dominated convergence. This shows continuity of $\tilde{N}$ as desired. □

3. Find all $q \geq 1$ such that $f(x^2) \in L_q((0,1), m)$ for any $f(x) \in L_4((0,1), m)$.

   Proof. If $q > 2$, then $f(x) = x^{-1/2q}$ is in L_4 but $f(x^2) = x^{-1/q}$ is not in L_q. Thus $q \leq 2$ for the given condition to hold. For $q = 2$, consider the function

$$f(x) = \frac{1}{x^{1/4}|\log(x/2)|^{1/2}}.$$

We claim that $f \in L_4$ but $f(x^2) \notin L_2$. Indeed

$$\int_0^1 f(x)^4 dx = \int_0^1 \frac{dx}{x \log(x/2)^2} = -\frac{1}{\log(x/2)}\bigg|_0^1 = \frac{1}{\log 2}.$$

while

$$\int_0^1 f(x^2)^2 dx = \int_0^1 \frac{dx}{x|\log(x^2/2)|} = -\frac{1}{2}\log|\log(x^2/2)|\bigg|_0^1 = -\frac{1}{2}\log\log 2 + \infty.$$

Finally, consider $1 \le q < 2$. Observe that by substitution $u = x^2$,

$$\int_0^1 f(x^2)^q dx = \int_0^1 \frac{f(u)^q}{2\sqrt{u}} du,$$

and the given bounds on q imply that $\frac{2}{3} \le \frac{2}{4-q} < 1$. By applying Holder with conjugate norms $p_1 = 4/q$ and $p_2 = 4/(4-q)$,

$$\begin{aligned}\int_0^1 \left|\frac{f(u)^q}{u^{1/2}}\right| du &\le \|f^q\|_{p_1} \cdot \|u^{-1/2}\|_{p_2} \\ &= \left(\int_0^1 f(u)^4 du\right)^{q/4} \left(\int_0^1 u^{-r} du\right)^{1-q/4} \quad \text{where } r = \frac{p_2}{2} = \frac{2}{4-q}\end{aligned}$$

and these last two integrals are finite. Thus $1 \le q < 2$ gives the desired range. □

4. Let

$$\begin{aligned} E &\subset \{(x,y) : 0 \le x \le 1,\, 0 \le y \le x\} \\ E_x &= \{y : (x,y) \in E\} \\ E^y &= \{x : (x,y) \in E\} \end{aligned}$$

and assume that $m(E_x) \ge x^3$ for any $x \in [0,1]$.

(a) Prove that there exists $y \in [0,1]$ such that $m(E^y) \ge \frac{1}{4}$.

(b) Prove that there exists $y \in [0,1]$ such that $m(E^y) \ge c$, where $c > 1/4$ is a constant independent of E. Give an explicit value of c.

Proof. (a) Suppose for a contradiction that $m(E^y) < 1/4$ for all $y \in [0,1]$. Then

$$\int_0^1 m(E^y) dy < \int_0^1 \frac{1}{4} dy = \frac{1}{4},$$

while by Fubini this integral is equal to

$$\int_0^1 m(E_x) dx \ge \int_0^1 x^3 dx = \frac{1}{4}.$$

This is a contradiction, so we must have $m(E^y) \ge 1/4$ for some y.

(b) Observe that

$$\int_0^1 m(E^y)dy \leq \int_0^{3/4} m(E^y)dy + \int_{3/4}^1 (1-y)dy = \int_0^{3/4} m(E^y)dy + \frac{1}{32}$$

since $E_y \subset \{x : y \leq x \leq 1\}$ so $m(E^y) \leq 1 - y$. As observed in part (a) we must have $\int_0^1 m(E^y)dy \geq 1/4$, so

$$\int_0^{3/4} m(E^y)dy \geq \frac{1}{4} - \frac{1}{32} = \frac{7}{32}.$$

This implies the "average value" of $m(E^y)$ on this interval is at least $\frac{7/32}{3/4} = 7/24$, so we may take $c = 7/24$ using the same argument as in part (a).

□

5. Let $E \subset [0,1]$ be a measurable set, $m(E) \geq \frac{99}{100}$. Prove that there exists $x \in [0,1]$ such that for any $r \in (0,1)$

$$m(E \cap (x-r, x+r)) \geq \frac{r}{4}.$$

Remark: One approach to this problem involves the Hardy-Littlewood maximal inequality

Proof. Let $f = \chi_{E^c}$, so the Hardy-Littlewood maximal function f^* is given by

$$f^*(x) = \sup_{x \in B} \frac{1}{m(B)} \int_B f(y)dy = \sup_{x \in B} \frac{m(B \cap E^c)}{m(B)} = 1 - \inf_{x \in B} \frac{m(B \cap E)}{m(B)}$$

where the supremum is taken over all balls (i.e. intervals) B containing x. It suffices to find some x satisfying $f^*(x) \leq 7/8$, since then

$$\inf_{r \in (0,1)} \frac{m(E \cap (x-r, x+r))}{2r} = \inf_{r \in (0,1)} \frac{m(B_r \cap E)}{m(B_r)} \geq \inf_{x \in B} \frac{m(B \cap E)}{m(B)} \geq \frac{1}{8},$$

where $B_r(x) = (x-r, x+r)$ denotes the ball of radius r around x.

The Hardy-Littlewood maximal inequality states that for any $\alpha > 0$,

$$m(x : f^*(x) > \alpha) \leq \frac{3}{\alpha} \|f\|_1.$$

In our case

$$\frac{3}{\alpha}\|f\|_1 = \frac{3}{\alpha}(1 - m(E)) \leq \frac{1}{\alpha} \frac{3}{100}.$$

Thus for $\alpha = 7/8$,

$$m(x : f^*(x) > 7/8) \leq \frac{8}{7} \cdot \frac{3}{100} < 1$$

so the complement $\{x : f^*(x) \leq 7/8\}$ has positive measure. In particular, it must be non-empty. □

Afternoon session

1. Find all entire functions $f(z)$ with the property that $g(z) \overset{\text{def}}{=} f(2z + \bar{z})$ is also entire.

   Proof. The Cauchy-Riemann equations state that

   $$\partial_x f(z) = f'(z),\ \partial_y f(z) = if'(z) \quad \Rightarrow \quad \partial_x f(z) = -i\partial_y f(z)$$

   for any holomorphic function f, where $z = x + iy$. If

   $$g(z) = g(x + iy) = f(2z + \bar{z}) = f(3x + iy),$$

   then

   $$\partial_x g(z) = 3f'(3x + iy) \quad \text{and} \quad -i\partial_y g(z) = f'(3x + iy),$$

   so g holomorphic at $x + iy$ implies $f' = 0$ at $3x + iy$. Thus if g is entire, this holds for all $3x + iy \in \mathbb{C}$ so $f' \equiv 0$. This means f must be constant. □

2. How many zeros does the polynomial

   $$p(z) = z^8 + 10z^3 - 50z + 1$$

   have in the right half-plane?

   Proof. First, observe that all eight zeros of p must lie inside the disk of radius two around the origin. Indeed, we may write $p(z) = f_1(z) + g_1(z)$ where

   $$f_1(z) = z^8 \quad \text{and} \quad g_1(z) = 10z^3 - 50z + 1$$

   and apply Rouche's theorem: on the circle $\{|z| = 2\}$ we have $|f_1(z)| = 256$ while $|g_1(z)| \leq 10|z|^3 + 50|z| + 1 = 181$, so p must have the same number of zeros inside this region as f_1, counting multiplicity, and this number is clearly 8 for f_1.

   Now let $p(z) = f_2(z) + g_2(z)$ where

   $$f_2(z) = z^8 - 50z + 1 \quad \text{and} \quad g_2(z) = 10z^3.$$

   On the imaginary axis $z = iy$,

   $$|g_2(iy)| = 10|y|^3 \quad \text{while} \quad |f_2(iy)| = |y^8 - 50iy + 1| \geq \max(y^8 + 1, 50|y|).$$

   Since

   $$10|y|^3 < y^8 + 1 \quad \text{on } [0, 1/3] \cup [2, \infty) \quad \text{and} \quad 10|y|^3 < 50|y| \quad \text{on } (0, 2],$$

   we see that $|g_2(z)| < |f_2(z)|$ along the imaginary axis. It is also straightforward to see this inequality holds on a circle of sufficiently large radius $R \geq 2$ by degree considerations. Thus for a sufficiently large semi-circular region in the right half-plane, Rouche's theorem implies that p has the same number of zeros in this region as f_2.

   We now consider the location of the zeros of $f_2(z) = z^8 - 50z + 1$, using Rouche's theorem a few more times. Write

   $$f_3(z) = z^8 - 50z, \quad g_3(z) = 1 \quad \text{so } f_2 = f_3 + g_3.$$

It is clear that $f_3(z) = z(z^7 - 50)$ has one zero at the origin and seven roots spaced evenly about the circle of radius $50^{1/7}$ around the origin, starting at the positive real axis. Thus f_3 has three roots in the right half-plane, four in the left half-plane, and one in the middle.

On the circle $\{|z| = 1/2\}$, we have $|f_3(z)| \geq 50|z| - |z|^8 > 20$ while $|g_3| = 1$, so $f_2 = f_3 + g_3$ and f_3 both have one zero inside this region.

Along the imaginary axis $\{iy : |y| \geq 1/2\}$ outside this circle, $|f_3(iy)| = |y^8 - 50iy| > 20 > |g_3|$, so both f_2 and f_3 must have three zeros in the region $\{z : \text{Re } z > 0, 1/2 < |z| < R\}$ for sufficiently large R.

It remains to decide whether f_2's zero near the origin is in the right or left half-plane. Considering f_2 as a function on the real line, we see that it must have a positive, real zero near the origin because it changes sign between $z = 0$ and $z = 1/2$:

$$f_2(0) = 1 > 0, \quad f_2(1/2) < -20 < 0 \quad \Rightarrow \quad f_2(x) = 0 \text{ for some } x \in (0, 1/2).$$

Thus f_2 contains four zeros in the right half-plane, as does p. □

3. Does there exist an analytic function f with an essential singularity at 0 such that $f(z)+2f(z^2)$ has a removable singularity?

Proof. We claim this is not possible. If f has an isolated singularity at 0, we may write the Laurent expansion of f around 0 as

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n = \cdots + a_{-1}z^{-1} + a_0 + a_1 z + \cdots.$$

Then $f(z) + 2f(z^2)$ has Laurent expansion

$$\begin{aligned} f(z) + 2f(z^2) &= \sum_{n=-\infty}^{\infty} b_n z^n = \sum_{n=-\infty}^{\infty} a_n z^n + \sum_{n=-\infty}^{\infty} 2a_n z^{2n} \\ &= \cdots + (a_{-2} + 2a_{-1})z^{-2} + a_{-1}z^{-1} + 3a_0 + a_1 z^1 + (a_2 + 2a_1)z^2 + \cdots. \end{aligned}$$

If this function has a removable singularity at 0, then all negative coefficients $\{b_n : n < 0\}$ in the Laurent expansion must vanish. Since $b_n = a_n$ for odd n, this means $a_n = 0$ for all odd $n < 0$. But for even coefficients,

$$b_{2n} = a_{2n} + 2a_n$$

so $b_{2n} = a_n = 0$ implies $a_{2n} = 0$. Thus all negative coefficients $\{a_n : n < 0\}$ must vanish, by induction on the highest power of 2 dividing n. This means f has a removable singularity at 0, *not* an essential singularity. □

4. Let $\{f_n : \mathbb{D} \to \mathbb{C}\}_{n=1}^{\infty}$ be a sequence of analytic functions such that $f_n(0) = 0$ for all $n \in \mathbb{N}$, and Re $f_n(z) \to 0$ uniformly on compact sets. Prove that Im $f_n(z) \to 0$ uniformly on compact sets.

Proof. Consider the sequence of analytic functions $g_n = e^{f_n} : \mathbb{D} \to \mathbb{C} - \{0\}$, which satisfy $g_n(0) = 1$. Since $|g_n| = e^{\mathrm{Re}\, f_n} \to e^0 = 1$ uniformly on compact sets, the sequence g_n is uniformly bounded in magnitude. We claim that the sequence g_n converges uniformly on compact sets to the constant function $g = 1$. If this were not true, then for some compact $K = \{|z| \leq 1 - \delta\} \subset \mathbb{D}$ and some $\epsilon > 0$ we could find an infinite subsequence g_{n_k} such that

$$\sup_{z \in K} |g_{n_k}(z) - 1| \geq \epsilon.$$

The family g_{n_k} is uniformly bounded on K, so by Montel's theorem there must be a sub(sub)sequence $g_{n_{k_j}}$ converging uniformly to an analytic function $\tilde{g}$. But since $|g_n| \to 1$ uniformly, $|\tilde{g}(z)| \equiv 1$ identically so $\tilde{g} = e^{i\theta}$ must be a constant on the unit circle (e.g. by the open mapping theorem). But $g_n(0) = 1$ for all n and $0 \in K$, so we must have $e^{i\theta} = 1$. Then $g_{n_{k_j}}$ in fact converges uniformly to 1, contrary to our initial assumption. This proves our claim that $g_n \to 1$ uniformly on compact subsets.

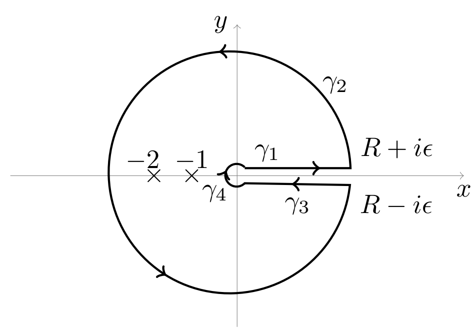
Now we claim that f_n must converge to 0, uniformly on compact sets. As before fix a compact set $K = \{|z| \leq 1 - \delta\} \subset \mathbb{D}$. For any $\epsilon > 0$, we can choose sufficiently large N such that

$$|g_n(z) - 1| < \epsilon \quad \text{for all } n \geq N,\ z \in K.$$

Namely, g_n must send K into on open ϵ-neighborhood of 1. The preimage of such a neighborhood under the map $z \mapsto e^z$, for ϵ sufficiently small, is contained in a disjoint union of small ϵ'-neighborhoods around the points $2\pi ik$, $k \in \mathbb{Z}$. Thus f_n must send K to such a disjoint union, but since K is connected, its image must lie in a single connected component. Moreover, it must lie in the component containing 0, since $f_n(0) = 0$. Thus f_n sends K to a small ϵ' neighborhood of 0 for sufficiently large n. As $\epsilon, \epsilon' \to 0$ this proves uniform convergence of $f_n \to 0$ as claimed. Thus $\mathrm{Im}\, f_n \to 0$ uniformly on compact sets. □

5. Use complex integration methods to compute $\displaystyle\int_0^\infty \frac{x^t}{(x+1)(x+2)} dx$, where $t \in (0, 1)$.

Proof. Consider integrating the holomorphic function $f(z) = \dfrac{z^t}{(z+1)(z+2)}$ over the "keyhole" contour shown below (the small circle has radius r):



We have

$$\lim_{\substack{\epsilon\to 0\\ r\to 0\\ R\to\infty}} \int_{\gamma_1} f(z)dz = \int_0^\infty \frac{x^t}{(x+1)(x+2)}dx =: I.$$

As we travel once around the origin to γ_3, recalling that $z^t := e^{t\log z}$, we gain an extra $2\pi i$ in the value of $\log z$ so

$$z^t = e^{t(\log|z|+2\pi i)} = |z|^t e^{2\pi i t}.$$

Thus

$$\lim_{\substack{\epsilon\to 0\\ r\to 0\\ R\to\infty}} \int_{\gamma_3} f(z)dz = \int_\infty^0 \frac{e^{2\pi i t}x^t}{(x+1)(x+2)}dx = -e^{2\pi i t} I.$$

We claim the integrals along γ_2 and γ_4 go to 0 as $R\to\infty$, $r\to 0$ respectively. Indeed

$$\left|\int_{\gamma_2} f(z)dz\right| \le |\gamma_2|\cdot\sup_{\gamma_2}|f(z)| \le 2\pi R\cdot\frac{R^t}{|R-1||R-2|} \to 0$$

as $R\to\infty$, since $t<1$. Similarly along γ_4

$$\left|\int_{\gamma_4} f(z)dz\right| \le |\gamma_4|\cdot\sup_{\gamma_4}|f(z)| \le 2\pi r\cdot\frac{r^t}{|1-r||2-r|} \to 0$$

as $r\to 0$.

Finally, we compute the residues of f at its poles, which occur at $z=-1$ and -2. We have

$$\text{Res}_f(-1) = \lim_{z\to -1}(z+1)f(z) = \frac{(-1)^t}{-1+2} = e^{\pi i t}, \quad \text{and}$$

$$\text{Res}_f(-2) = \lim_{z\to -2}(z+2)f(z) = \frac{(-2)^t}{-2+1} = -2^t e^{\pi i t}.$$

By Cauchy's integral formula

$$(1-2^t)e^{\pi i t} = \sum \text{Res}_f = \frac{1}{2\pi i}\lim_{\substack{\epsilon\to 0\\ r\to 0\\ R\to\infty}} \int_{\gamma_1+\gamma_2+\gamma_3+\gamma_4} f(z)dz = \frac{1}{2\pi i}(1-e^{2\pi i t})I,$$

which shows $I = 2\pi i\dfrac{(1-2^t)e^{\pi i t}}{1-e^{2\pi i t}} = \dfrac{\pi}{\sin(\pi t)}(2^t-1)$. □