Morning session

1. Let f be a continuously differentiable function on $\mathbb{R}$. Assume there exist constants $a, b \in \mathbb{R}$ such that
$$f(x) \to a, \quad f'(x) \to b \quad \text{as } x \to \infty.$$
Prove or give a counterexample: b must be zero.

Proof. We claim b must be zero. Indeed for any $\epsilon > 0$ we may choose N such that
$$|f(x) - a| < \epsilon \quad \text{and} \quad |f'(x) - b| < \epsilon \quad \text{for all } x \geq N.$$
Then for any $M > 0$
$$\left| \int_N^{N+M} f'(x) dx \right| = |f(N+M) - f(N)| < 2\epsilon$$
and
$$\left| \int_N^{N+M} (b - f'(x)) dx \right| \leq \int_N^{N+M} |f'(x) - b| dx < M\epsilon.$$
while the sum of these integrals is $\displaystyle\int_N^{N+M} b\, dx = Mb$. Thus $|b| < \epsilon + 2\epsilon/M$, so as $\epsilon \to 0$ this implies $b = 0$ as claimed. □

2. Using techniques of real analysis (as opposed to complex analysis) show that
$$\lim_{R\to\infty} \int_0^R \frac{\sin(x)}{x} dx = \frac{\pi}{2}.$$

Proof. Observe that
$$\int_0^\infty e^{-xt} dt = \frac{1}{x},$$
so applying Fubini's theorem we have
$$\begin{aligned}\lim_{R\to\infty} \int_0^R \frac{\sin(x)}{x} dx &= \lim_{R\to\infty} \int_0^R \left(\int_0^\infty e^{-xt} dt \right) \sin(x) dx \\ &= \int_0^\infty \left(\lim_{R\to\infty} \int_0^R e^{-xt} \sin(x) dx \right) dt.\end{aligned}$$
To evaluate the inner integral we use integration by parts:
$$\begin{aligned}\int_0^\infty e^{-xt} \sin(x) dx &= -e^{-xt} \cos(x) \Big|_0^\infty - t \int_0^\infty e^{-xt} \cos(x) dx \\ &= 1 - te^{-xt} \sin(x) \Big|_0^\infty - t^2 \int_0^\infty e^{-xt} \sin(x) dx \\ &= 1 - t^2 \int_0^\infty e^{-xt} \sin(x) dx\end{aligned}$$

so

$$\lim_{R\to\infty}\int_0^R e^{-xt}\sin(x)dx = \frac{1}{1+t^2}.$$

Then

$$\lim_{R\to\infty}\int_0^R \frac{\sin(x)}{x}dx = \int_0^\infty \left(\frac{1}{1+t^2}\right)dt = \arctan(t)\Big|_0^\infty = \frac{\pi}{2}.$$

□

3. Let f be a measurable function on a measure space (X,μ), where μ is a finite measure. Suppose there are constants $K>0$ and $p>1$ such that

$$\mu\{x\in X : |f(x)|>M\} < \frac{K}{M^p} \quad \text{for all } M>0.$$

Prove that f is integrable.

Proof. For any $M>0$ we may write the above expression as an integral of the characteristic function

$$\int_X \chi(M<|f(x)|)dx = \mu\{x\in X : |f(x)|>M\} < \frac{K}{M^p},$$

and by assumption that μ is a finite measure we also have the trivial bound

$$\int_X \chi(M<|f(x)|)dx \le \mu(X).$$

By application of Tonelli's theorem

$$\begin{aligned}\int_X |f(x)|dx &= \int_X\int_0^\infty \chi(y<|f(x)|)dydx\\ &= \int_0^\infty\int_X \chi(y<|f(x)|)dxdy\\ &\le \int_0^1 \mu(X) + \int_1^\infty \frac{K}{y^p}dy = \mu(X) + \frac{K}{p-1} < \infty.\end{aligned}$$

Thus f is integrable. □

4. Let E be a measurable subset of $[0,1]$. Assume there is a constant $\alpha>0$ such that

$$m(E\cap I) \ge \alpha m(I) \quad \text{for all intervals } I\subset[0,1].$$

Prove that $m(E)=1$.

Proof. Since $m(I) = m(I\backslash E) + m(E\cap I)$ the above condition implies that

$$m(I\backslash E) \le (1-\alpha)m(I) \quad \text{for all intervals } I.$$

Consider the complement E^c, which is measurable since E is. By definition of the Lebesgue measure, for any $\epsilon > 0$ we can find a finite collection of disjoint intervals I_k such that

$$E^c \subset \bigsqcup_k I_k \quad \text{and} \quad m(E^c) \leq \sum_k m(I_k) \leq m(E^c) + \epsilon.$$

Then we have $E^c = \bigsqcup_k (I_k \cap E^c) = \bigsqcup_k (I_k \backslash E)$, so

$$m(E^c) = \sum_k m(I_k \backslash E) \leq (1-\alpha) \sum_k m(I_k) \leq (1-\alpha) m(E^c) + (1-\alpha)\epsilon.$$

As $\epsilon \to 0$, this implies $0 \leq -\alpha m(E^c)$ so we must have $m(E^c) = 0$. Thus $m(E) = 1$ as desired. □

5. Let $(f_n)_{n=1}^\infty$ be a sequence of non-negative measurable functions on a measure space (X, μ), where μ is a finite measure. Assume that f_n converges almost everywhere to an integrable function f.
   (a) Show by example that in general $\lim_{n\to\infty} \int f_n d\mu$ may be infinite.
   (b) Suppose $\lim_{n\to\infty} \int f_n d\mu = \int f d\mu$. Prove that $f_n \to f$ in L_1, that is

$$\lim_{n\to\infty} \int |f_n - f| d\mu = 0.$$

Proof. (a) Consider the sequence of functions $f_n : [0,1] \to \mathbb{R}$ (with the Lebesgue measure) defined by

$$f_n(x) = \begin{cases} n^2 & \text{if } 0 < x < 1/n \\ 0 & \text{otherwise,} \end{cases}$$

which converges pointwise to the function $f = 0$. We have

$$\int_{[0,1]} f_n d\mu = n \to \infty \quad \text{as } n \to \infty.$$

(b) For each $n \geq 1$, define $g_n = \max(f_n - f, 0)$ and $h_n = \min(f_n - f, 0)$ so that

$$f_n - f = g_n + h_n \quad \text{and} \quad |f_n - f| = g_n - h_n = |g_n| + |h_n|.$$

It is clear that both sequences g_n and h_n converge to 0 a.e. Since f_n and f are non-negative, $|h_n| \leq |f|$ so by dominated convergence

$$\lim_{n\to\infty} \int |h_n| d\mu = \int \lim_{n\to\infty} |h_n| d\mu = 0.$$

Then since $\int |f_n - f| d\mu = \int |g_n| d\mu + \int |h_n| d\mu$, it suffices to show that $\int |g_n| d\mu \to 0$. But we are given that

$$\lim_{n\to\infty} \int (f_n - f) d\mu = \lim_{n\to\infty} \int (g_n + h_n) d\mu = 0,$$

so $\lim\limits_{n\to\infty} \int g_n d\mu = \lim\limits_{n\to\infty} \int |g_n| d\mu = 0$ as desired.

□

Afternoon session

1. Let Δ denote the unit disk $\{z \in \mathbb{C} : |z| < 1\}$, let H denote the right half-plane $\{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ and let f be an analytic function mapping H into Δ and satisfying $f(5) = 0$. Based on the given information, what are you able to say about $f'(5)$?

   Proof. Observe that the möbius transformation
$$g(z) = \frac{z-5}{z+5}$$
is an analytic isomorphism from H to Δ sending 5 to 0, such that
$$g'(5) = \frac{10}{(5+5)^2} = \frac{1}{10}.$$
We may factor the given $f\colon H \to \Delta$ as a composition $h \circ g$, (namely $h = f \circ g^{-1}\colon \Delta \to \Delta$) so that $h(0) = 0$ and $f'(5) = h'(g(5))g'(5) = h'(0)/10$. By Schwarz-Pick
$$|h'(z)| \leq \frac{1-|h(z)|^2}{1-|z|^2} \quad \Rightarrow \quad |h'(0)| \leq 1 \quad .$$
Thus we have $|f'(5)| \leq 1/10$.

   For any $\alpha \in \overline{\Delta} = \{z : |z| \leq 1\}$, the function $f = \alpha g$ satisfies the given hypotheses, and has $f'(5) = \alpha/10$. So the above condition is the only constraint on the value $f'(5)$. $\square$

2. Let $f = u+iv$ be an entire function with the property that $(u^2)_{xx} + (u^2)_{yy}$ vanishes identically along the real axis. (The subscripts denote partial derivatives.) What can you conclude about f?

   Proof. Along the real axis, we have
$$\begin{aligned}(u^2)_{xx} + (u^2)_{yy} &= (2uu_x)_x + (2uu_y)_y = 2(uu_{xx} + (u_x)^2) + 2(uu_{yy} + (u_y)^2)\\ &= 2u(u_{xx} + u_{yy}) + 2(u_x^2 + u_y^2) = 0.\end{aligned}$$
Since f is entire, the real and imaginary parts u, v are both harmonic, so $u_{xx} + u_{yy} = 0$ on the entire domain. Thus the above equality implies
$$(u_x)^2 + (u_y)^2 = 0 \quad \Rightarrow \quad u_x = u_y = 0$$
along the real axis, since both u_x and u_y are real-valued. Thus u is constant along the real axis, and by Cauchy-Riemann ($v_x = -u_y = 0$) v is also constant. Thus f is constant along the real axis, so its unique analytic continuation to the complex plane is constant as well. $\square$

3. Find all solutions of $\cos(z) = 1 + 100z^2$ in the unit disk $|z| < 1$.

Proof. Note that finding solutions to the given equation is equivalent to finding zeros of the function $f(z) = 100z^2 + 1 - \cos(z)$. Taking the Taylor expansion of $\cos(z)$ around 0, we have

$$f(z) = 100.5z^2 + (\text{higher-order terms})$$

so f has a zero of multiplicity 2 at $z = 0$.

This is in fact the only zero of f in the unit disk, which we prove using Rouche's theorem. For z in the closed unit disk we have in particular that $|\text{Im } z| \leq 1$, so

$$|1 - \cos(z)| \leq 1 + |\cos(z)| \leq 1 + |\cosh(\text{Im } z)| \leq 1 + \frac{e + e^{-1}}{2} < 3,$$

while $|100z^2| = 100$ on the boundary of the disk. Thus we have the strict inequality

$$|1 - \cos(z)| < |100z^2|$$

on the boundary of this region, so $f(z)$ and $100z^2$ must have the same number of zeros, counted with multiplicity, in the interior. This zero-count is clearly 2 for the function $100z^2$, so we have already found all zeros of f as claimed. □

4. Let S denote the strip $\{z = x + iy : 0 \leq y \leq 1\}$. Suppose that f is analytic in a neighborhood of S and satisfies $|f(x + iy)| \leq \frac{C}{1+x^2}$ on S for some constant C.

   (a) Show that
   $$I(t, y) \stackrel{\text{def}}{=} \int_{\mathbb{R}} f(x + iy)e^{-ixt}dx$$
   is a well-defined bounded continuous function of $(t, y) \in \mathbb{R} \times [0, 1]$.

   (b) Under what conditions will $|I(t, 1)|$ be bigger than $|I(t, 0)|$?

Proof. (a) For any choice of t, y in the given range, the integrand $f(x+iy)e^{-ixt}$ is continuous and thus locally integrable. By assumption on the decay of $|f|$,

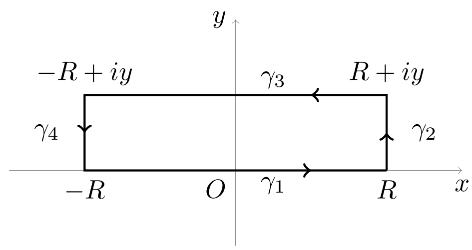
$$|I(t, y)| \leq \int_{\mathbb{R}} |f(x + iy)e^{-ixt}|dx \leq \int_{\mathbb{R}} \frac{C}{1 + x^2}dx = C\pi.$$

Thus the integral defining $I(t, y)$ converges and is bounded in magnitude by the constant $C\pi$. To see that this function is continuous, for any sequence $(\epsilon_n, \delta_n) \to (0, 0)$

$$\begin{aligned}\lim_{n\to\infty} |I(t + \epsilon_n, y + \delta_n) - I(t, y)| &\leq \lim_{n\to\infty} \int_{\mathbb{R}} |f(x + i(y + \delta_n))e^{-ix(t+\epsilon_n)} - f(x + iy)e^{-ixt}|dx \\ &= \int_{\mathbb{R}} \lim_{n\to\infty} |f(x + i(y + \delta_n))e^{-ix\epsilon_n} - f(x + iy)|dx = 0,\end{aligned}$$

(by continuity of f) where we use dominated convergence to justify moving the limit inside the integral, as the integrand is dominated by the integrabale function $\dfrac{2C}{1 + x^2}$.

(b) Consider integrating $g(z) = f(z)e^{-itz}$ along the closed contour indicated below:



It is clear that $I(t,0) = \lim_{R\to\infty} \int_{\gamma_1} g(z)dz$. Along γ_3, we have

$$\int_{\gamma_3} g(z)dz = \int_R^{-R} f(x+iy)e^{-it(x+iy)}dx = -e^{ty}\int_{-R}^{R} f(x+iy)e^{-itx}dx,$$

so $\lim_{R\to\infty}\int_{\gamma_3} g(z)dz = -e^{ty}I(t,y)$. Finally, over γ_2

$$\left|\int_{\gamma_2} g(z)dz\right| \le y \cdot \sup_{\gamma_2} |g(z)| \le \frac{C}{1+R^2} \to 0 \quad \text{as } R\to\infty$$

and the same bound holds for the integral along γ_4. Since $g(z)$ is analytic within this contour for any R, Cauchy's integral formula tells us

$$0 = \lim_{R\to\infty}\int_{\gamma_1+\gamma_2+\gamma_3+\gamma_4} g(z)dz = I(t,0) - e^{ty}I(t,y).$$

Thus for $y=1$ we have $I(t,0) = e^t I(t,1)$, so $|I(t,1)|$ will be larger when $t<0$.

$\square$

5. Construct a function $f(z)$ analytic for $0<|z|<1$ so that $e^{f(z)}$ has a pole at $z=0$, or else explain why no such function exists.

Proof. We explain that this is not possible. If $g(z) = e^{f(z)}$ were to have a pole at 0, say of order $k\ge 1$, then by the argument principle

$$\int_{|z|=r} \frac{g'(z)}{g(z)}dz = 2\pi i(\{\#\text{ of zeros}\} - \{\#\text{ of poles}\}), \quad \text{for any } r<1$$

where the zeros and poles of g are counted inside the region $\{|z|<r\}$ with multiplicity. For sufficiently small r, the only zero or pole in this region will occur at $z=0$ so the above integral will evaluate to $-2\pi i k \neq 0$. However, as $g(z) = e^{f(z)}$ we also have

$$\int_{|z|=r} \frac{g'(z)}{g(z)}dz = \int_{|z|=r} \frac{f'(z)e^{f(z)}}{e^{f(z)}}dz = \int_{|z|=r} d(f(z)) = 0,$$

for any r. Thus $e^{f(z)}$ cannot have a pole at the origin. $\square$