Morning session

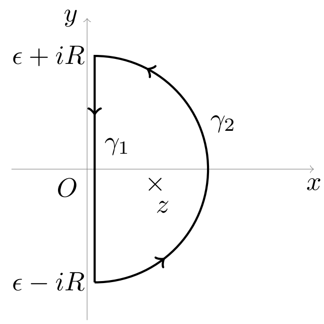
1. Let $\mathbb{H} = \{z \in \mathbb{C} : \text{Re }(z) > 0\}$, and let $f : \bar{\mathbb{H}} \to \mathbb{C}$ be a bounded continuous function, which is analytic in $\mathbb{H}$. Prove that for any $z = x + iy \in \mathbb{H}$

$$f(z) = \frac{x}{\pi} \int_{-\infty}^{\infty} \frac{f(it)dt}{x^2 + (t-y)^2}.$$

Proof. Consider integrating the function

$$g(w) = \frac{f(w)}{(w+\bar{z})(w-z)}$$

over the semicircular contour indicated below:



This function is meromorphic inside the given region, and has a single pole at $w = z$ with residue

$$\text{Res}_g(z) = \lim_{w \to z} (w-z)g(z) = \frac{f(z)}{z + \bar{z}} = \frac{f(z)}{2x}.$$

Along γ_2, we have

$$\left| \int_{\gamma_2} g(w)dw \right| \le |\gamma_2| \cdot \sup_{\gamma_2} |g(w)|$$
$$\le \pi R \cdot \frac{M}{(R - |z|)^2} \approx \frac{\pi M}{R} \to 0 \quad \text{as } R \to \infty.$$

where M is a uniform bound on $|f(w)|$ in $\bar{\mathbb{H}}$. Along γ_1, we have

$$\lim_{\epsilon \to 0} \int_{\gamma_1} g(w)dw = \lim_{\epsilon \to 0} \int_{R}^{-R} \frac{f(\epsilon + it)}{(\epsilon + it + \bar{z})(\epsilon + it - z)} idt$$
$$= -i \int_{-R}^{R} \frac{f(it)dt}{(x + i(t-y))(-x + i(t-y))} = i \int_{-R}^{R} \frac{f(it)dt}{x^2 + (t-y)^2},$$

where passing the limit inside the integral is justified by dominated convergence under $\dfrac{M}{(x/2)^2 + (t-y)^2}$ (assuming $\epsilon < x/2$).

Thus Cauchy's integral formula tells us that

$$\frac{f(z)}{2x} = \sum \mathrm{Res}_g = \lim_{\substack{\epsilon\to 0\\ R\to\infty}} \frac{1}{2\pi i}\int_{\gamma_1+\gamma_2} g(w)dw = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{f(it)dt}{x^2+(t-y)^2},$$

and multiplying the above equation by $2x$ gives the desired result. □

2. Let $f\colon \mathbb{D}\to\mathbb{D}$, $f(z)=\sum_{n=0}^{\infty} a_n z^n$, be a bounded analytic function.

   (a) Prove that for any $r<1$

$$\sum_{n=0}^{\infty} |a_n|^2 r^{2n} = \frac{1}{2\pi}\int_0^{2\pi} |f(re^{it})|^2 dt.$$

   (b) Show that the series $\sum_{n=0}^{\infty} |a_n|^2$ converges.

Proof. (a) Let $f_N = \sum_{n=0}^{N} a_n z^n$ denote the degree-N Taylor approximation of f. Note that for any $r<1$, the convergence $f_N\to f$ is uniform on the circle $\{|z|=r\}$ since f is analytic in $\mathbb{D}$. We have for any $z=re^{it}\in\mathbb{D}$

$$|f_N(z)|^2 = (\sum_{n=0}^{N} a_n z^n)(\sum_{n=0}^{N} \bar{a}_n \bar{z}^n) = \sum_{n=0}^{N} |a_n|^2|z|^{2n} + \sum_{0\le n\ne m\le N} a_n\bar{a}_m z^n\bar{z}^m$$
$$= \sum_{n=0}^{N} |a_n|^2 r^{2n} + \sum_{n\ne m} a_n\bar{a}_m r^{n+m} e^{(n-m)it}.$$

It is straightforward to verify that integrating z^k or $\bar{z}^k$ around a circle gives you zero:

$$\int_0^{2\pi} (re^{it})^k dt = \int_0^{2\pi} (re^{-it})^k dt = 0 \quad \text{for } k\ne 0,$$

(by orthogonality of $\cos(kt), \sin(kt)$) while the other terms are constant. Thus

$$\frac{1}{2\pi}\int_0^{2\pi} |f_N(re^{it})|^2 dt = \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{n\le N} |a_n|^2 r^{2n}\right)dt + \sum_{n\ne m}\left(a_n\bar{a}_m r^{n+m}\frac{1}{2\pi}\int_0^{2\pi} e^{(n-m)it}dt\right)$$
$$= \sum_{n\le N} |a_n|^2 r^{2n}$$

Taking $N\to\infty$ gives the desired equality.

(b) As r approaches 1, the uniform bound $|f(z)|\le 1$ ensures that

$$\lim_{r\to 1}\frac{1}{2\pi}\int_0^{2\pi} |f(re^{it})|^2 dt \le \frac{1}{2\pi}\int_0^{2\pi} dt = 1.$$

Thus

$$\sum_{n\geq 0}|a_n|^2 = \lim_{r\to 1}\left(\sum_{n\geq 0}|a_n|^2 r^{2n}\right) \leq 1$$

is finite by monotone convergence.

□

3. Let f be a function which is analytic on the wedge

$$W = \{z \in \mathbb{C} : \text{Re } (z) > 0, -\frac{\pi}{6} < \text{Arg } (z) < \frac{\pi}{6}\},$$

which is bounded on W, and verifies for all $r > 0$

$$\lim_{\theta\to\pm\frac{\pi}{6}} f(re^{i\theta}) := \varphi(r) \in \mathbb{R}.$$

Show that f must be real and constant.

Hint: Consider using Schwarz reflection.

Proof. Following the hint, we note that f may be extended to an entire function by reflecting its values on W six times around the origin. Since f is bounded on W, this extension to $\mathbb{C}$ will also be bounded. Thus f must be constant, and since it takes real values (in some limit) it must be a real constant. □

4. Evaluate

$$\int_0^\infty \frac{\ln x dx}{(x-1)\sqrt{x}}.$$

Proof. Denote the given integral by I. Substituting $y = \ln(x)$

$$I = \int_{-\infty}^{\infty} \frac{ye^y dy}{(e^y - 1)e^{y/2}} = \int_{-\infty}^{\infty} \frac{y\, dy}{e^{y/2} - e^{-y/2}}.$$

If we consider integrating $f(z) = \dfrac{z}{e^{z/2} - e^{-z/2}}$ along the rectangular contour that encloses $\{z : 0 \leq \text{Im } z \leq \pi, -R \leq \text{Re } z \leq R\}$, which does not contain any poles of f, we see that the integrals over the vertical segments go to 0 as $R \to \infty$, so the integral along the lower boundary must equal the integral along the upper boundary (taking both integrals from left to right). Thus

$$\begin{aligned} I = \int_{-\infty}^{\infty} \frac{(y+\pi i)dy}{e^{(y+\pi i)/2} - e^{-(y+\pi i)/2}} &= \int_{-\infty}^{\infty} \frac{y}{ie^{y/2} + ie^{-y/2}} dy + \int_{-\infty}^{\infty} \frac{\pi i}{ie^{y/2} + ie^{-y/2}} dy \\ &= -i \int_{-\infty}^{\infty} \frac{y}{e^{y/2} + e^{-y/2}} dy + \pi \int_{-\infty}^{\infty} \frac{1}{e^{y/2} + e^{-y/2}} dy \\ &= \pi \int_{-\infty}^{\infty} \frac{1}{e^{y/2} + e^{-y/2}} dy \end{aligned}$$

since the integrand in the first term is an odd function.

Substituting back $w = e^{y/2}$,

$$\int_{-\infty}^{\infty} \frac{1}{e^{y/2} + e^{-y/2}} dy = \int_{-\infty}^{\infty} \frac{e^{y/2} dy}{e^y + 1} = \int_0^{\infty} \frac{2dw}{w^2 + 1} = 2 \arctan(w) \Big|_0^{\infty} = \pi.$$

Thus $I = \pi^2$. □

5. Let $\Omega \subset \mathbb{C}$ be a bounded, simply connected domain in $\mathbb{C}$. Let z_0 and z_1 be two distinct points of Ω. If φ_1 and φ_2 are two one-to-one and onto analytic maps from Ω onto itself, and $\varphi_1(z_i) = \varphi_2(z_i)$, $i = 0, 1$, show that $\varphi_1 \cong \varphi_2$ on Ω.

Proof. The given conditions of Ω imply that it is analytically isomorphic to the unit disk $\mathbb{D}$ by the Riemann mapping theorem. Thus we may take some analytic isomorphism $\Phi\colon \Omega \to \mathbb{D}$, which we may assume sends $z_0 \mapsto 0$ without loss of generality. Then $\tilde{\varphi}_1 := \Phi \circ \varphi_1 \circ \Phi^{-1}$ defines a bijective analytic map from $\mathbb{D}$ to itself, as does $\tilde{\varphi}_2 := \Phi \circ \varphi_2 \circ \Phi^{-1}$.

Now consider $\tilde{\varphi}_2^{-1} \circ \tilde{\varphi}_1\colon \mathbb{D} \to \mathbb{D}$ which is equal to $\Phi \circ (\varphi_2^{-1} \circ \varphi_1) \circ \Phi^{-1}$. This fixes the origin and the point $w_1 := \Phi(z_1)$. In particular, this means $|\tilde{\varphi}_1(w_1)| = |w_1|$, so Schwarz's lemma implies that $\tilde{\varphi}_2^{-1} \circ \tilde{\varphi}_1$ is the identity (i.e. the only rotation that fixes w_1). Then $\varphi_2^{-1} \circ \varphi_1 = \Phi^{-1} \circ (\tilde{\varphi}_2^{-1} \circ \tilde{\varphi}_1) \circ \Phi = \Phi^{-1} \circ \Phi$ must be the identity on Ω, so $\varphi_1 \equiv \varphi_2$. □

Afternoon session

1. Let $f \in L^1((0,1))$, and define $g\colon (0,1) \to \mathbb{R}$ by

$$g(x) = \int_x^1 \frac{f(t)}{t} dt.$$

Prove that $g \in L^1((0,1))$.

Proof. We apply Tonelli's theorem to exchange the order of integration:

$$\begin{aligned} \int_0^1 |g(x)| dx &\leq \int_0^1 \int_x^1 \frac{|f(t)|}{t} dt dx \\ &= \int_0^1 \frac{|f(t)|}{t} \int_0^t dx dt = \int_0^1 |f(t)| dt. \end{aligned}$$

Thus $\|g\|_1 \leq \|f\|_1 < \infty$ by assumption $f \in L^1$, so $g \in L^1$ as well. □

2. Let $(X, \mathcal{A}, \mu)$ be a finite measure space, and let $F\colon \mathbb{R} \to \mathbb{R}$ be a C^2 function with second derivative $F'' > 0$. Let $f \in L^1(\mu)$ be real-valued. Prove *Jensen's inequality*:

$$F\left(\frac{1}{\mu(X)} \int f d\mu\right) \leq \frac{1}{\mu(X)} \int F(f) d\mu.$$

Proof. For any $t_0, t \in \mathbb{R}$,

$$\begin{aligned} F(t) &= F(t_0) + \int_{t_0}^t F'(s) ds = F(t_0) + (t - t_0) F'(t_0) + \int_{t_0}^t \int_{t_0}^s F''(r) dr ds \\ &\geq F(t_0) + (t - t_0) F'(t_0) \end{aligned}$$

since $F'' > 0$ and the two integrals are either both positively oriented, or both negatively oriented. (Moreover, the inequality is *strict* if $t \neq t_0$.)

Now take $f_0 = \frac{1}{\mu(X)} \int f d\mu \in \mathbb{R}$, so that

$$\int_X f d\mu = \mu(X) f_0 = \int_X f_0 \, d\mu \quad \Rightarrow \quad \int_X (f - f_0) d\mu = 0.$$

We have $F(f) \geq F(f_0) + (f - f_0) F'(f_0)$ for any $f \in \mathbb{R}$, so integrating this inequality over X we have

$$\int_X F(f) d\mu \geq F(f_0) \int_X d\mu + F'(f_0) \int_X (f - f_0) d\mu = \mu(X) F(f_0).$$

Dividing by $\mu(X)$ gives the desired result. □

3. Let $f, g_1, g_2, \ldots \in L^1(\mathbb{R})$ be non-negative functions. Assume that $g_n \to f$ a.e. and

$$\int_{\mathbb{R}} g_n dm = \int_{\mathbb{R}} f dm.$$

Prove that for any measurable set $A \subseteq \mathbb{R}$

$$\int_A g_n dm \to \int_A f dm.$$

Proof. For any measurable A, Fatou's lemma says that

$$\int_A f dm = \int_A (\liminf_{n\to\infty} g_n) dm \leq \liminf_{n\to\infty} \int_A g_n dm.$$

To prove $\lim_{n\to\infty} \int_A g_n = \int_A f$ it suffices to show that $\int_A f \geq \limsup_{n\to\infty} \int_A g_n$. For this, consider integrating these over the complement A^c:

$$\int_{A^c} f dm = \int_{A^c} (\liminf_{n\to\infty} g_n) dm \leq \liminf_{n\to\infty} \int_{A^c} g_n dm.$$

Expressing these in terms of the total integral over $\mathbb{R}$ gives

$$\int_{\mathbb{R}} f dm - \int_A f dm \leq \liminf_{n\to\infty} \left(\int_{\mathbb{R}} g_n dm - \int_A g_n dm \right) = \int_{\mathbb{R}} f dm - \limsup_{n\to\infty} \int_A g_n dm$$

so we must have

$$\int_A f dm \geq \limsup_{n\to\infty} \int_A g_n dm.$$

This shows the desired convergence of integrals. □

4. Let $(X, \mathcal{A}, \mu)$ be a finite measure space. Let $\{f_n\}_{n=1}^\infty \subset L^2(\mu)$ be a sequence of functions such that $\|f_n\|_2 \leq 1$.

(a) Prove that if $f_n \to 0$ in measure, then $f_n \to 0$ in $L^1(\mu)$.

(b) If $f_n \to 0$ in measure, does it necessarily follow that $f_n \to 0$ in $L^2(\mu)$?

Proof. (a) Suppose $f_n \to 0$ in measure, meaning that for any $\epsilon > 0$

$$\lim_{n\to\infty} \mu(\{x : |f_n(x)| > \epsilon\}) = 0.$$

The Cauchy-Schwarz inequality implies that for any measurable $A \subset X$,

$$\int_A |f_n| d\mu \leq \left(\int_A |f_n|^2 d\mu \right)^{1/2} \left(\int_A d\mu \right)^{1/2} \leq (\mu(A))^{1/2}$$

since we are given that $\|f\|_2 \leq 1$.

Now consider the measurable sets $E_{n,\epsilon} = \{x : |f_n(x)| > \epsilon\}$. By convergence in measure

$$\limsup_{n\to\infty} \int_{E_{n,\epsilon}} |f_n| d\mu \leq \limsup_{n\to\infty} \mu(E_{n,\epsilon})^{1/2} = 0$$

for any fixed $\epsilon > 0$, so

$$\begin{aligned}\limsup_{n\to\infty} \int_X |f_n| d\mu &= \limsup_{n\to\infty} \left(\int_{E_{n,\epsilon}^c} |f_n| d\mu + \int_{E_{n,\epsilon}} |f_n| d\mu \right) \\ &\le \limsup_{n\to\infty} \int_{E_{n,\epsilon}^c} \epsilon \, d\mu + \limsup_{n\to\infty} \int_{E_{n,\epsilon}} |f_n| d\mu \le \epsilon \cdot m(X).\end{aligned}$$

As $\epsilon \to 0$, this bound goes to zero so $f_n \to 0$ in L^1 as desired.

(b) No; consider the sequence in $L^2([0,1])$ with the Lebesgue measure defined by

$$f_n(x) = \begin{cases} \sqrt{n} & \text{if } 0 < x < \frac{1}{n}, \\ 0 & \text{otherwise.} \end{cases}$$

It is straightforward to verify that $f_n \to 0$ in measure, but $\|f_n\|_2 = 1$ for all n so $f_n \not\to 0$ in L^2.

□

5. Let $F \subset \mathbb{R}$ be a closed set, and define the distance from $x \in \mathbb{R}$ to F by

$$d(x, F) = \inf_{y \in F} |x - y|.$$

Prove that

$$\lim_{x \to y} \frac{d(x, F)}{|x - y|} = 0$$

for a.e. $y \in F$.

Hint: Consider Lebesgue points of F.

Proof. Recall that $y \in F$ is a Lebesgue point of F if

$$\lim_{\substack{m(B)\to 0 \\ y \in B}} \frac{m(B \cap F)}{m(B)} = 1 \quad \Leftrightarrow \quad \lim_{\substack{m(B)\to 0 \\ y \in B}} \frac{m(B) - m(B \cap F)}{m(B)} = 0$$

where the limit is taken over balls (i.e. intervals) B in $\mathbb{R}$ that contain y. Since Lebesgue points have full measure in F it suffices to prove the given equality for Lebesgue points.

Suppose y is a Lebesgue point of F, and $x \in \mathbb{R}$ arbitrary. Let $B_r(x)$ denote the ball of radius r centered at x, and let $B_r := B_r(x, y)$ denote the smallest open interval containing both $B_r(x)$ and $B_r(y)$. It is clear that $m(B_r) = |x - y| + 2r$. If we intersect this ball with F, then we must exclude an interval of length at least $d(x, F)$:

$$m(B_r \cap F) \le m(B_r) - d(x, F) \quad \Rightarrow \quad d(x, F) \le m(B_r) - m(B_r \cap F).$$

This implies

$$\frac{d(x, F)}{|x - y|} \le \liminf_{r \to 0} \frac{m(B_r) - m(B_r \cap F)}{|x - y|} = \liminf_{r \to 0} \frac{m(B_r) - m(B_r \cap F)}{m(B_r)}.$$

Since y is a Lebesgue point, for any $\epsilon > 0$ there is an $\delta >$ such that

$$m(B) < \delta,\, y \in B \quad \Rightarrow \quad \left| \frac{m(B) - m(B \cap F)}{m(B)} \right| < \epsilon.$$

so taking $B = B_r(x, y)$ for any x satisfying $0 < |x - y| < \delta$, we have

$$0 \leq \frac{d(x, F)}{|x - y|} \leq \liminf_{r \to 0} \frac{m(B_r) - m(B_r \cap F)}{m(B_r)} < \epsilon.$$

In the limit as x approaches y, we get $\lim_{x \to y} \frac{d(x, F)}{|x - y|} < \epsilon$. But since ϵ was arbitrary, this shows that the limit is in fact 0. □